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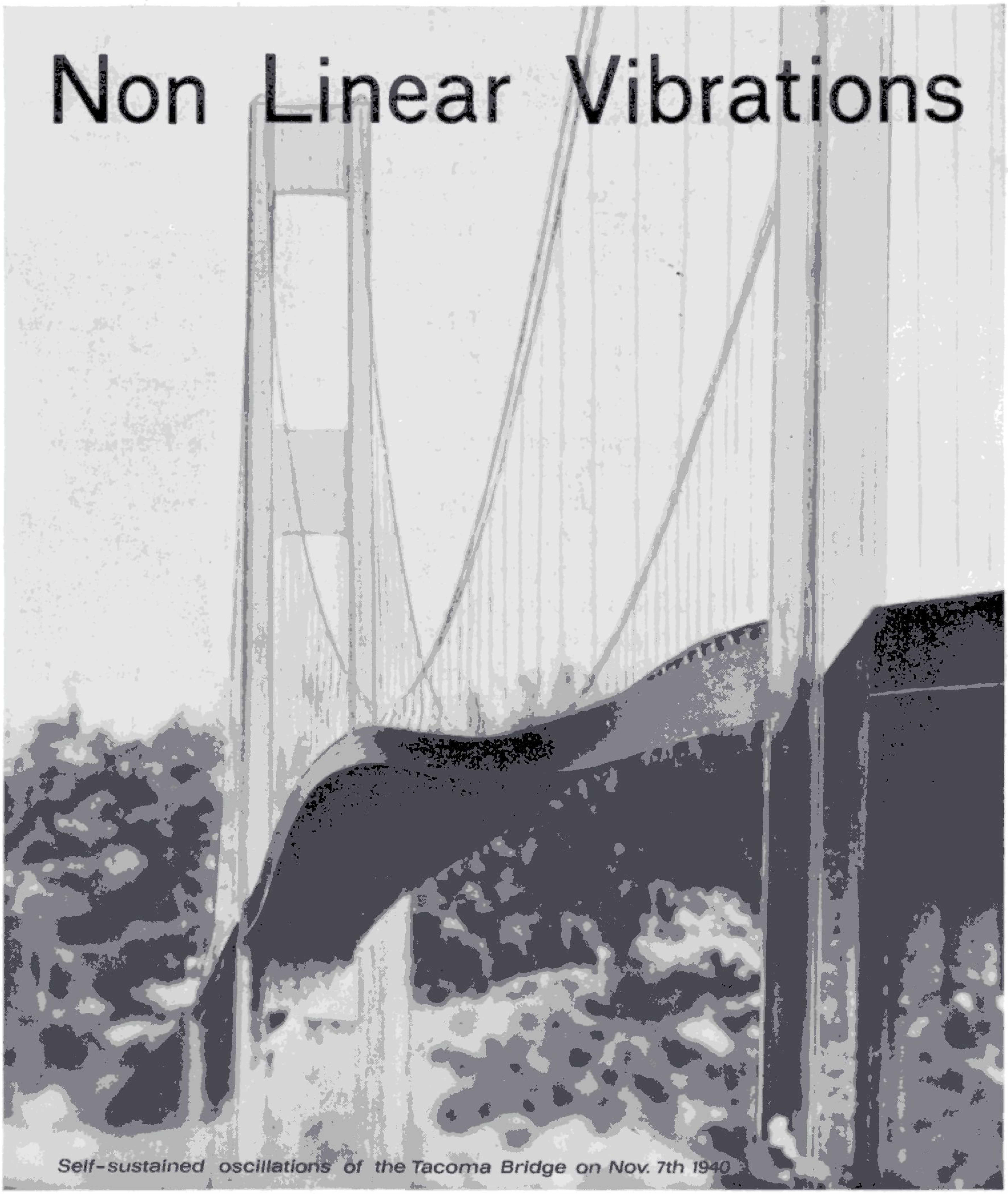

Self-sustained oscillations of the Tacoma Bridge on Nov. 7th 1940

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Non-linear Amplitude Distortion in Vibrating Systems

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ABSTRACT

The various effects produced by non-linear, frequency dependent systems are briefly outlined and examples of such systems described. To demonstrate some of the effects of non-linearity, a number of experiments have been carried out on electrical analogue circuits. Measurements were made on a "hardening" spring system, a "softening" spring system, systems with velocity dependent, positive damping and on a two-degrees-of-freedom system. Response curves, frequency spectra and wave-shapes are shown, and it is found that among the configurations being studied the "hardening" spring system seems to be the most "dangerous" one with respect to the production of harmonics. Furthermore, systems with velocity dependent damping where the damping increases greatly with the level of excitation are found to be sources of potential damage due to the change in their transmissibility characteristics.

Finally, the problems involved in controlling vibration exciters loaded by non-linear test specimens are briefly discussed.

SOMMAIRE

Les différentes propriétés des systèmes non-linéaires et sensibles à la fréquence sont brièvement rappelées et des exemples de tels systèmes sont décrits. La démonstration de certains des effets de la non-linéarité a été effectuée expérimentalement sur circuits analogues électriques. Les mesures portèrent sur des systèmes élastiques à rigidité augmentante ou diminuante, sur des systèmes à amortissement positif fonction de la vitesse, ainsi que sur un système à deux degrés de liberté. Les courbes de réponse, spectres de fréquence et formes d'ondes obtenus sont montrés, et on en déduit qu' entre les configurations mises à l'étude le systême élastique «durcissant» est le plus apte à produire des harmoniques dangereuses. D'autre part, les systêmes à amortissement dépendent de la vitesse dont l'amortissement augmente rapidement avec le niveau d'excitation peuvent être à l'origine de troubles en raison des variations de leur caractéristique de transmission.

En conclusion, les problèmes relatifs au contrôle des excitateurs de vibration chargés par des structures présentant des non-linéarités sont brièvement discutés.

ZUSAMMENFASSUNG

An Hand von Beispielen werden die Eigenschaften nicht-linearer frequenzabhängiger Systeme aufgezeigt. Elektrische Analogschaltungen dienen zur Demonstration. Untersucht werden Federn mit zunehmender und abnehmender Steife, Systemen mit positiver Dämpfung und Systemen mit zwei Freiheitsgraden. Die Ergebnisse enthalten Wiedergabekurven, Frequenzspektren und Kurvenformen. Die zunehmende Steife ist ungünstig, weil sie Oberwellen hervorruft. Wächst die geschwindigkeitsabhängige Dämpfung in starkem Masse mit dem Anregungspegel, so können sich die Übertragungseigenschaften in schädlicher Weise ändern. Abschliessend wird die Reglung nichtlinear belasteter Schwingerreger kurz diskutiert.

Introduction.

It is well known that the output from a sinusoidally excited linear system will also be sinusoidal (as long as the system contains some finite damping). Also it is well known that the output from a system containing one or more amplitude non-linearities will consist of a series of harmonically related sinusoids when the system is excited sinusoidally. However, the magnitude of the various harmonics depends upon the nature of the non-linearity. If the non-linearity is symmetrical around zero only odd harmonics of the input signal will occur in the output.

Futhermore, if the non-linear system is frequency independent (shows a "flat" amplitude vs. frequency response) the relationship between the harmonics will remain the same, independent of the frequency of the input sinusoid. The latter is normally the case in the non-linear system produced by overdriving an electronic amplifier. However, it is normally *not* the case in non-linear mechanical (or electrical) resonance systems.
In the following some symmetrical non-linear systems occurring in engineering practice will be investigated and the effects of the non-linearity upon the response discussed.

Some Examples of Non-linear Systems.

Let it be stated at once that amplitude non-linearities occur almost everywhere in practice to a greater or lesser degree. Even the higher quality amplifiers produce harmonic distortions of the order of 0.01 to 0.1 % depending upon the magnitude of the input signal. (If the input signal consists of a complex wave with statistically distributed peaks (such as music, speech or noise) some of the peaks will always be high enough to "overdrive" the amplifier).
On the other hand, very small non-linearities might not be of great significance when judging the output signal from the amplifier and the system may in general be classified as "linear". Quite another picture exists in the vibration study of complicated mechanical structures. If some non-linear element produces harmonics of the order of 0.1 to 1 % these might excite resonances somewhere in the structure and cause severe vibration amplitudes to occur "unexpectedly". Say, for example, that a non-linear spring-element produces a third harmonic of the order of 1 %, and the frequency of the harmonic coinsides with a resonance with a Q of 100, this specific resonance will vibrate with the same amplitude as the main vibration, even though its resonance frequency did not exist in the input signal! This is even more important as a "hardening" spring*), or non-linear damping, is sometimes used on purpose by designers to "damp" a specific resonance.
It might be worth while at this stage also to point out some other properties of frequency dependent non-linear systems. If, for example, in a spring-mass resonance system (single degree-of-freedom system) *the spring is non-linear* the response of the system to a sinusoidal input will not only contain a number of harmonics, but *the relative magnitude of the harmonics will also depend upon the frequency of the input signal.*
Futhermore, *the resonance frequency will depend upon the excitation level.* This is readily seen from the equation governing the resonance frequency $f = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$, where "c" is the spring constant and "m" the resonating mass. In the case of a "hardening" spring, "c" will increase with excitation

*) A "hardening" spring is a spring which becomes "stiffer" by deflection

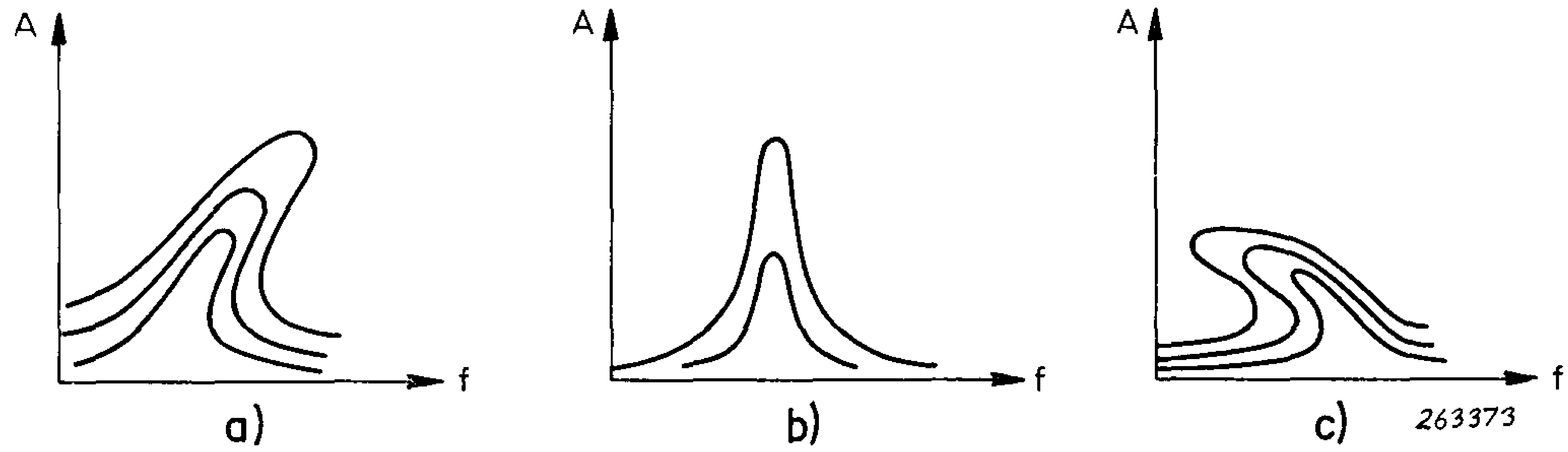


Fig. 1. Typical resonance curves for various levels of excitation for:
a) A "hardening" spring type resonant system.
b) A linear resonant system.
c) A "softening' spring type resonant system.

amplitude, thus causing the resonance to move towards higher frequencies. Conversely, if "c" decreases with excitation amplitude the resonance will move towards lower frequencies and the spring is said to have a "softening" characteristic. Fig. 1 illustrates how the resonance frequency depends upon the spring characteristics and the excitation level.

A great variety of non-linear spring characteristics exists in practice, such as gradually "hardening" or "softening" springs, set-up springs, springs which limit the deflection amplitude etc. However, in general these may be treated as specific, limiting cases of either a gradually "hardening" or a "softening" spring system.

If the *non-linearity is situated in the damping* element of a resonating system a somewhat different situation exists. Also in this case the production of harmonics varies with frequency and excitation level, but the resonance frequency itself remains practically constant. A special case occurs when the damping is negative in that in this case the system oscillates. A practical example is the vacuum-tube (or transistor) oscillator. Also in mechanical systems these kind of self-sustained oscillations may take place e.g. the flutter of aeroplane wings, oscillations in electrical transmission lines due to the action of the wind, and some cases of Columb friction. One of the most disastrous examples of damage caused by non-linear behaviour is the failure of the Tacoma Bridge in the USA.

Finally, if the non-linearity of a resonant system is "strong" enough and the "effective" damping small enough, *the system may produce so-called sub-harmonics.* A sub-harmonic is a vibration occurring at 1/2, 1/3, 1/4, 1/5 etc. of the exciting frequency. The occurrence of subharmonics in practice is relatively rare, but examples of systems where such vibrations may well be found are the well-known electro-acoustic transducer — the loudspeaker, and the parametric amplifier. Sub-harmonic vibrations were analyzed by Prof. P. O. Pe-

dersen of the Danish Technical University in 1933 and physically an explanation for their occurrence may be given in that the exciting signal supplies energy to one of the harmonics of the non-linear system. When energy is supplied the system will start to oscillate and the higher harmonic will pull all the other harmonics with it, as the specifically excited harmonic is an integral part of the whole motion.
An excellent survey of non-linear periodic vibrations is given in the last chapter of the book "Mechanical Vibration" by J. P. Den Hartog.

Methods of Solving Non-linear Vibration Problems.

In the preceding text effects produced by non-linear spring and damping elements in vibrating systems were briefly described. No mention, however, was made of systems with non-linear masses. The reason for this is that non-linear effective masses *may* occur in practice but have up to now not seemed to be of any general interest. Also, a number of conclusions on their effects can be drawn from the study of other "reactive" elements.
Mathematically the solving of non-linear differential equations governing non-linear vibrating systems exactly may only be possible in a few very special cases. To treat the problem more generally approximate methods must be used, which either limit the solution to "small non-linearities" or present complicated computation schemes. The use of computers can, however, in many cases furnish more or less complete solutions for particular cases.
The intention of this article is not to "dig deeply" into the theory of non-linear mathematics, but much more on an experimental basis to show the effects of non-linear vibratory systems in the form of frequency response curves, wave-forms and frequency spectra.
Use is made of electrical-mechanical analogies and measurements performed on simple electrical circuits. For example in the so-called "mobility"-analogue the spring element of a mechanical system is "substituted" by an inductance. Non-linear inductances are very simple to produce by furnishing the coil with an iron core, and when the iron-core coil is connected in series with a linear capacitor a non-linear resonant system of the "hardening" spring type is obtained for "large" vibratory amplitudes. A disadvantage of the iron core coil (at least for some exactly analogue purposes) may be that it shows typical hysteresis effects and will give a spring characteristic which, for very small vibratory amplitudes shows a slightly "softening" tendency until it, for lager amplitudes, becomes of the typical "hardening" spring type.
By using two iron-core coils with opposite magnetic bias together with a capacitor it is possible to obtain a resonant system with "softening" spring characteristics within certain limits. This will be further explained later in the article. Finally, by the use of V.D.R.'s (Voltage Dependent Resistors) velocity-dependent damping characteristics of a resonant system can be

readily produced. Also, if some sudden amplitude changes like those produced by mechanical stops, are present in the vibrating system, these effects can be accounted for in the electrical analogue with the aid of switching diodes.

Basic Analogue Circuits.

As mentioned previously a great variety of non-linear mechanical systems exists in practice. However, to illustrate the effects of the non-linearity measurements on some basic systems may suffice, and in the following two arrangements, a single-degree-of-freedom system and a two-degrees-of-freedom system, will be outlined.

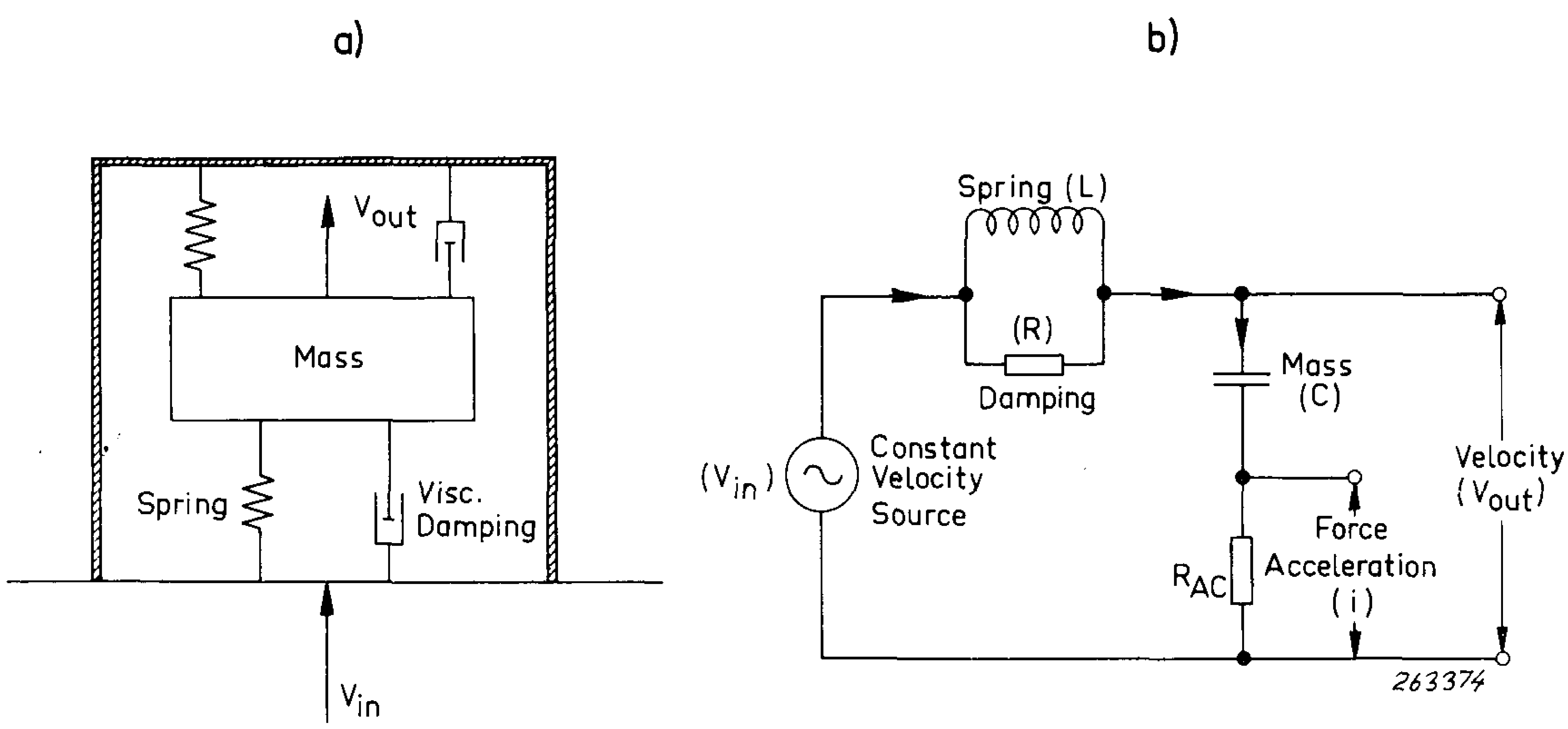


Fig. 2. Single degree-of-freedom mechanical system and its electrical "mobility" analogue.
a) The mechanical system
b) The electrical analogue.

a) Single degree-of-freedom system.

A single-degree-of-freedom resonant system consists of a mass, a spring and some sort of damping. Two important cases exist in practice: The mass can be the mass of a heavy machine, supported by flexible mounts on the "floor" of a mechanical workshop, etc. in which case the exciting forces operate *directly* on the mass, — or the mass can be some sort of an instrument, heavy article of furniture etc. mounted on a vibrating "floor" in which case the exciting forces operate on the mass *through the supports.*

It can be shown that the same type of differential equation governs the relative motion of the mass in both cases, although the exciting force function must be "chosen" differently.

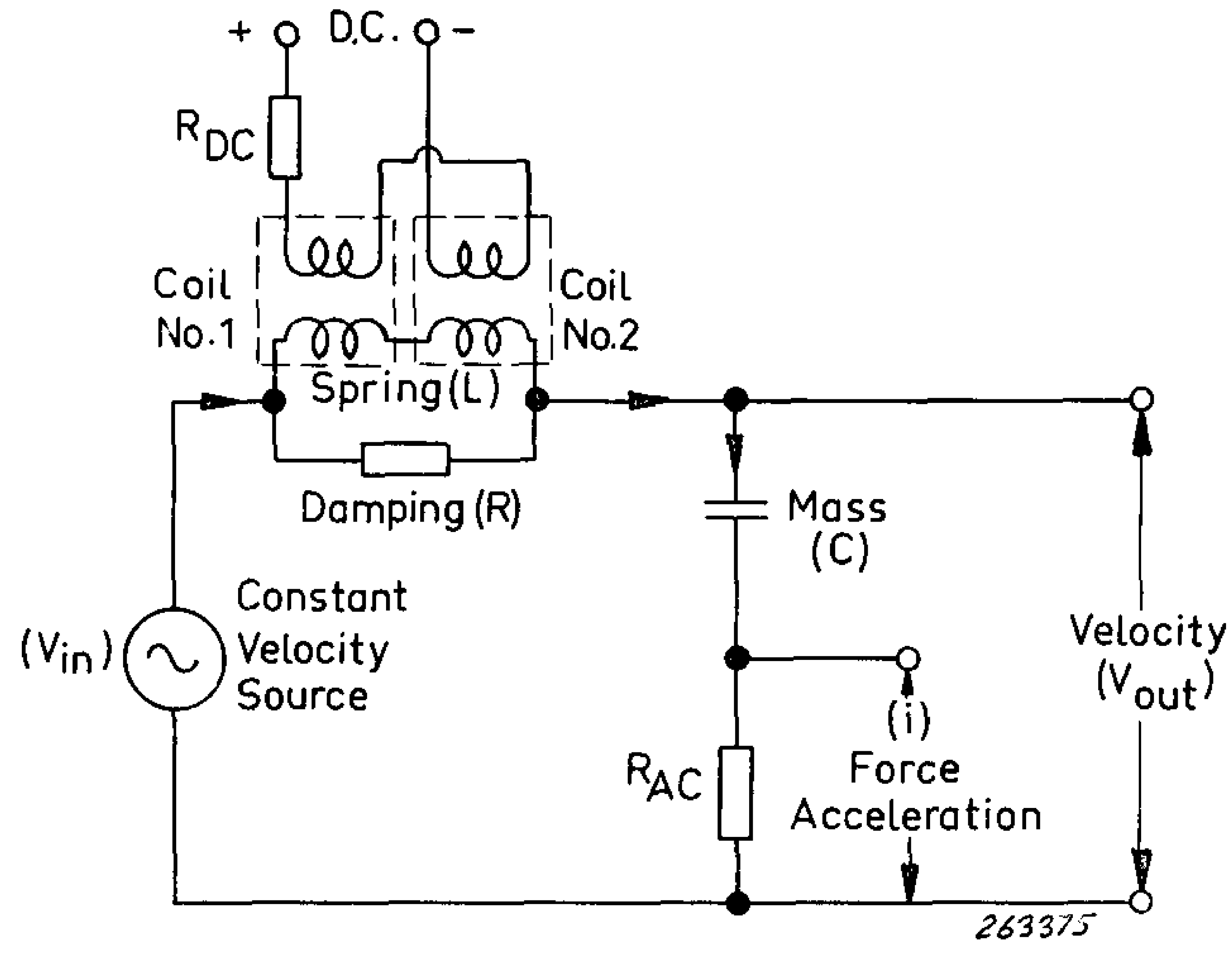


Fig. 3. Electrical circuit used to produce non-linearity in the "spring"-element (inductance).

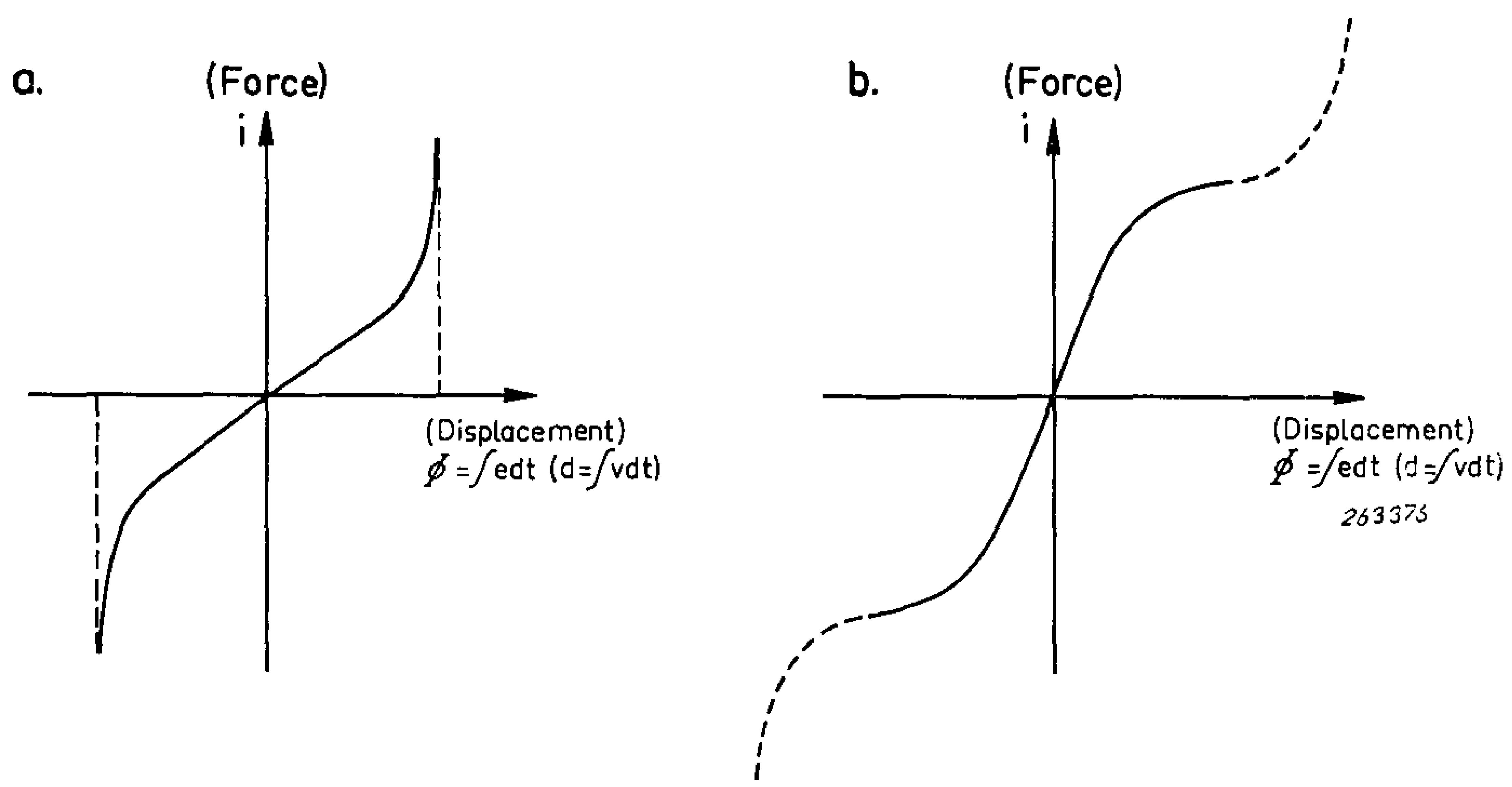


Fig. 4. Basic operating characteristics of the non-linear "spring" element. The relative displacement of the spring is represented by the voltage integral ∫edt and the operating characteristic is "equal" to the magnetizing curve for the iron core inductor .(The magnetizing curves have been replotted in a fashion which is commonly used in mechanical engineering for plotting force vs. displacement curves).

a) Typical "hardening" spring characteristic.

b) Typical "softening' spring characteristic. (Only the portion of the curve drawn in full was used for the experiments).

Only the latter case, i.e., the case where the force operates on the mass through the suspension system is considered here. This system is shown in Fig. 2a while the basic electrical "mobility" analogue of the system is given in Fig. 2b.
Fig. 3 shows the actual electrical circuit used to produce a vibrating system of either the "hardening" spring type or the "softening" spring type.
If no D.C. current flows through the second windings of the coils the system will be of the "hardening" spring type operating on the magnetizing characteristic of the iron core, see Fig. 4a. The coils are simply connected in series and act as one coil. When D.C. current flows it will magnetize the coils in opposite direction, and if the coils are exactly equal the total operating characteristic will be of the type shown in Fig. 4b. The resistor marked R_{DC} in Fig. 3 must be high enough to minimize the losses and the hysteresis effect on the characteristic Fig. 4b. Actual operating curves obtained for the circuit used in the experiments are shown in Fig. 5.

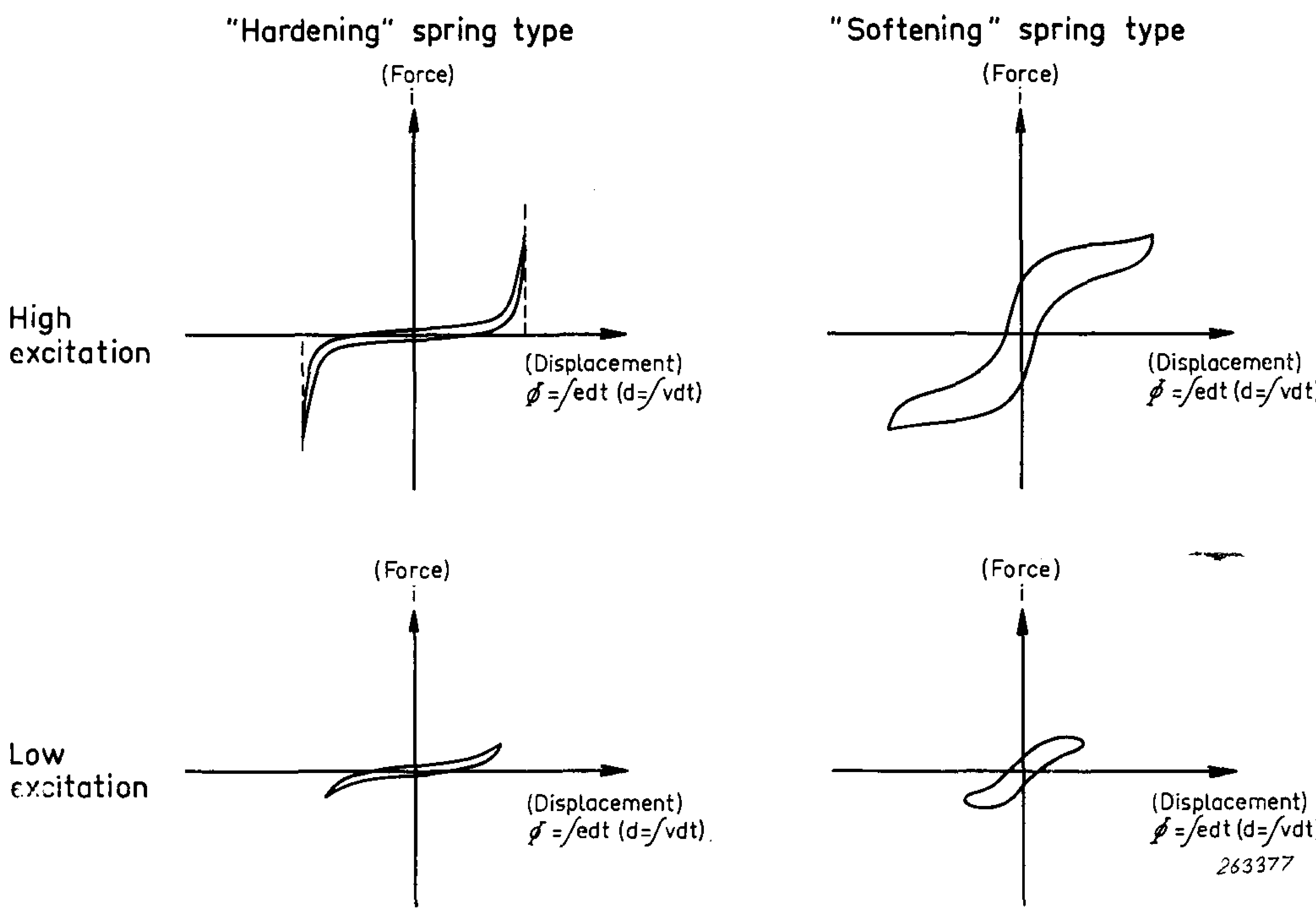
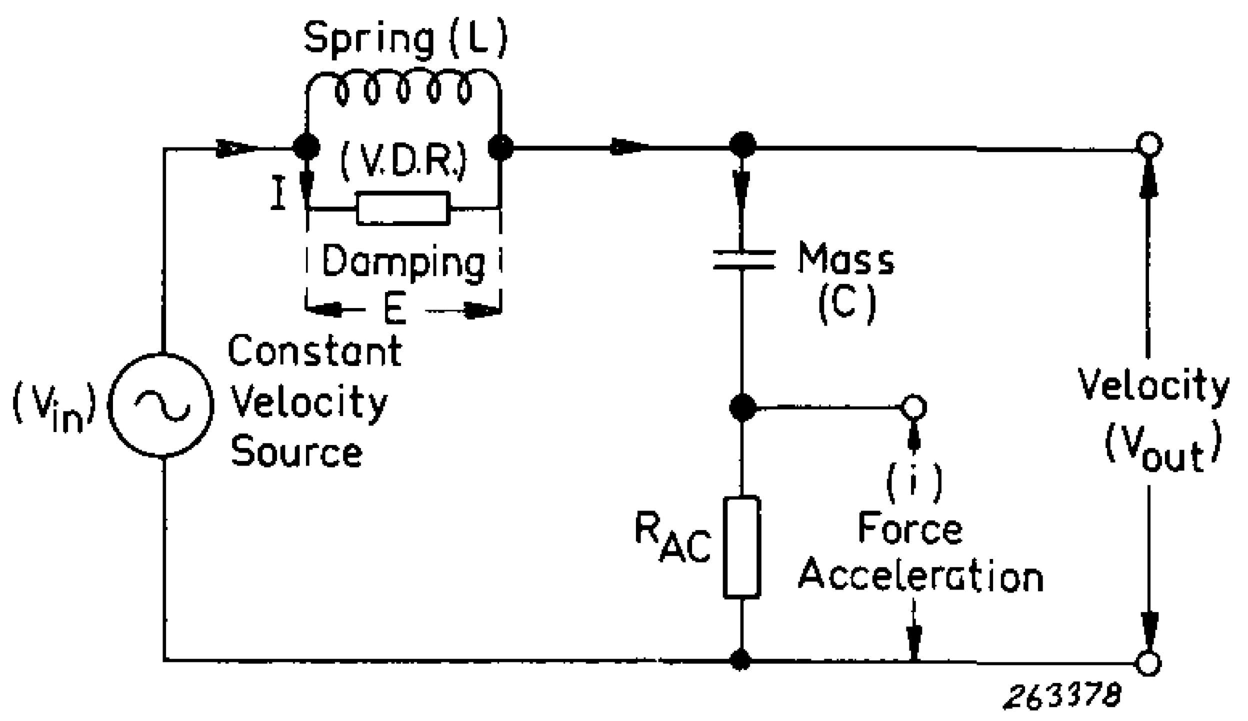
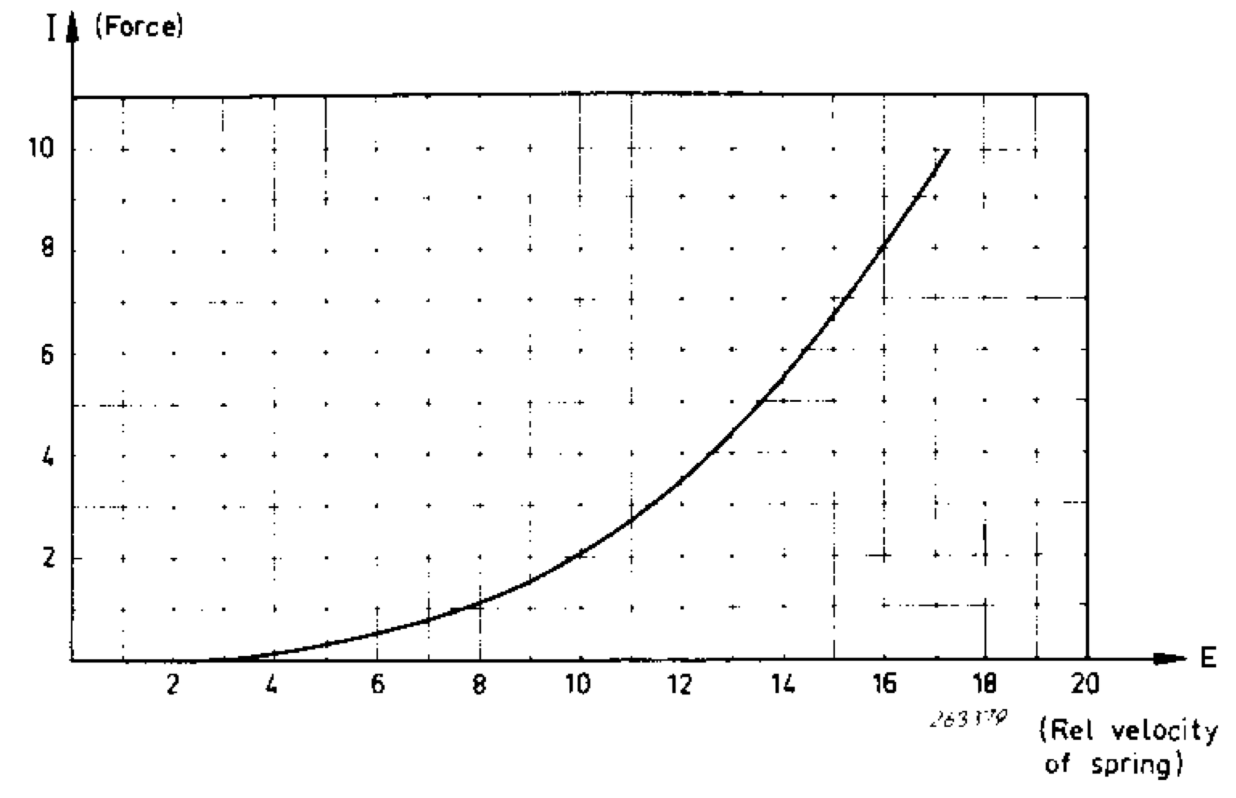


Fig. 5. Actual operating characteristics. The curves were here plotted from the screen of an oscilloscope.



a)



b)

Fig. 6. Electrical analogue circuit used for experiments with velocity dependent damping.
a) Electrical circuit.
b) Typical operating curve for the V.D.R.

Fig. 6a shows the circuit used to produce a velocity-dependent damping characteristic, and in Fig. 6b the operating curve of the V.D.R. is shown.
In the "mobility"-analogue, the voltage across the capacitor symbolizes the velocity of the mass. Thus the current through the capacitor $\left(i_c = C \frac{de_c}{dt} \right)$ will symbolize the acceleration of the mass, and to be able to study this acceleration a *very small* resistor, marked R_{ac} in Figs. 3 and 6a has been introduced.

b) Two degrees-of-freedom system.

The two-degrees-of-freedom system which will be studied here is shown in Fig. 7a while Fig. 7b shows the electrical analogue to the system. Only the first of the two coupled circuits is considered to be non-linear. By the introduction of a second resonating system a number of new parameters become involved, f. inst. the ratio between the two resonance frequencies and the degree-of-coupling between the two circuits. To eliminate the effect of coupling in the first instance, an electronic amplifier (B & K Type 2409) was introduced between the two circuits. The amplifier must show a flat frequency and phase response in the frequency range of interest in order not to upset the result, and the response of Type 2409 is shown in Fig. 8. It can be seen that this amplifier (Voltmeter) fulfils the requirements in the range from some 40 c/s to around 10ooo c/s.

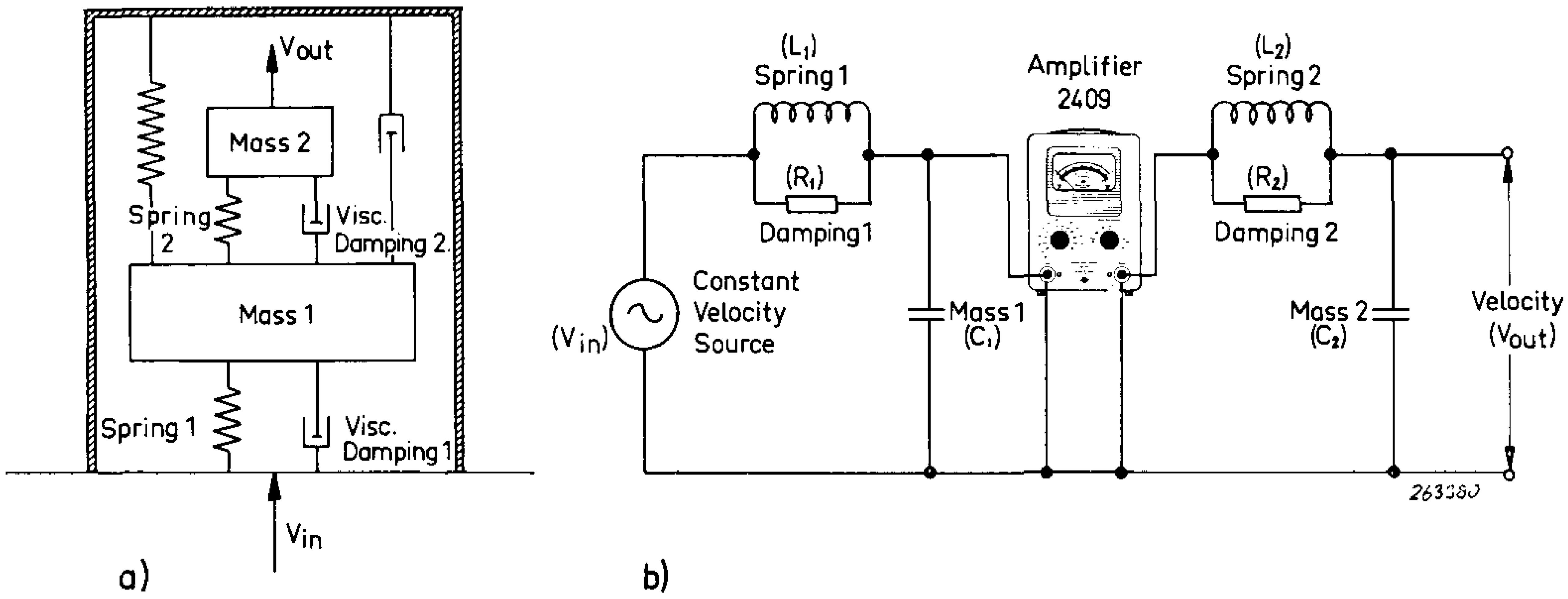


Fig. 7. Two degrees-of-freedom system with a "small" mass elastically supported on a "large" mass (negligible coupling between the masses)
a) The mechanical system.
b) The electrical analogue.

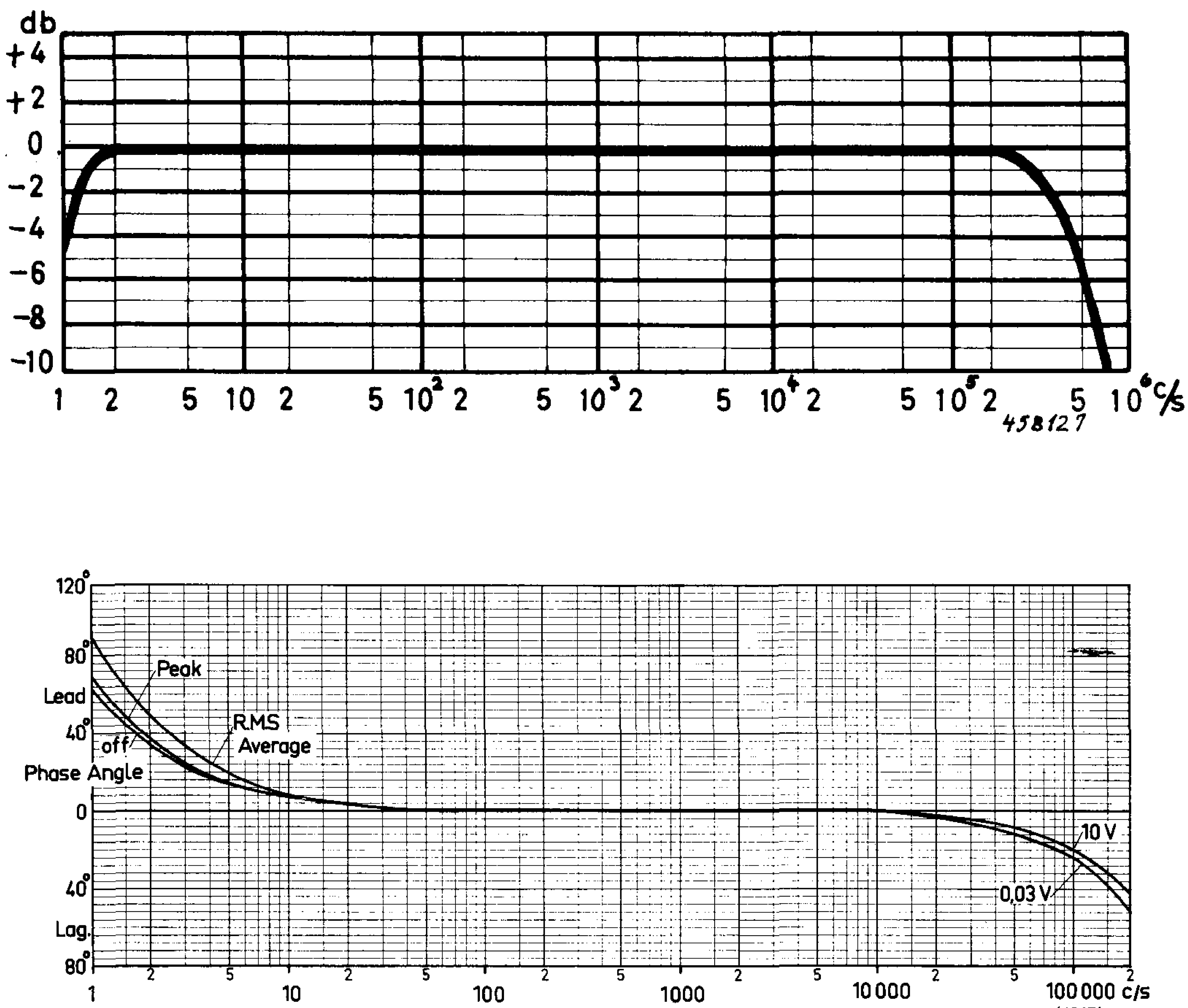


Fig. 8. Typical frequency response and phase shift characteristics of the Voltmeter Type 2409.

Single Degree-of-Freedom Systems.

It is convenient, for the sake of clarity, to study each of the three systems considered, the "hardening" spring system, the "softening" spring system and the system with velocity dependent damping separately.

a) The "hardening" spring system.

This system is very important in practice and is often incorporated in vibration isolation installations in order to reduce the maximum displacements of a specific mass. However, the wave-shapes and spectra produced by a "hardening" spring system contain a great number of significant harmonics which may cause great deflections at higher frequencies elsewhere in the supported structure. It is therefore necessary when applying a "hardening" spring in a vibration isolation problem to take these effects into account.

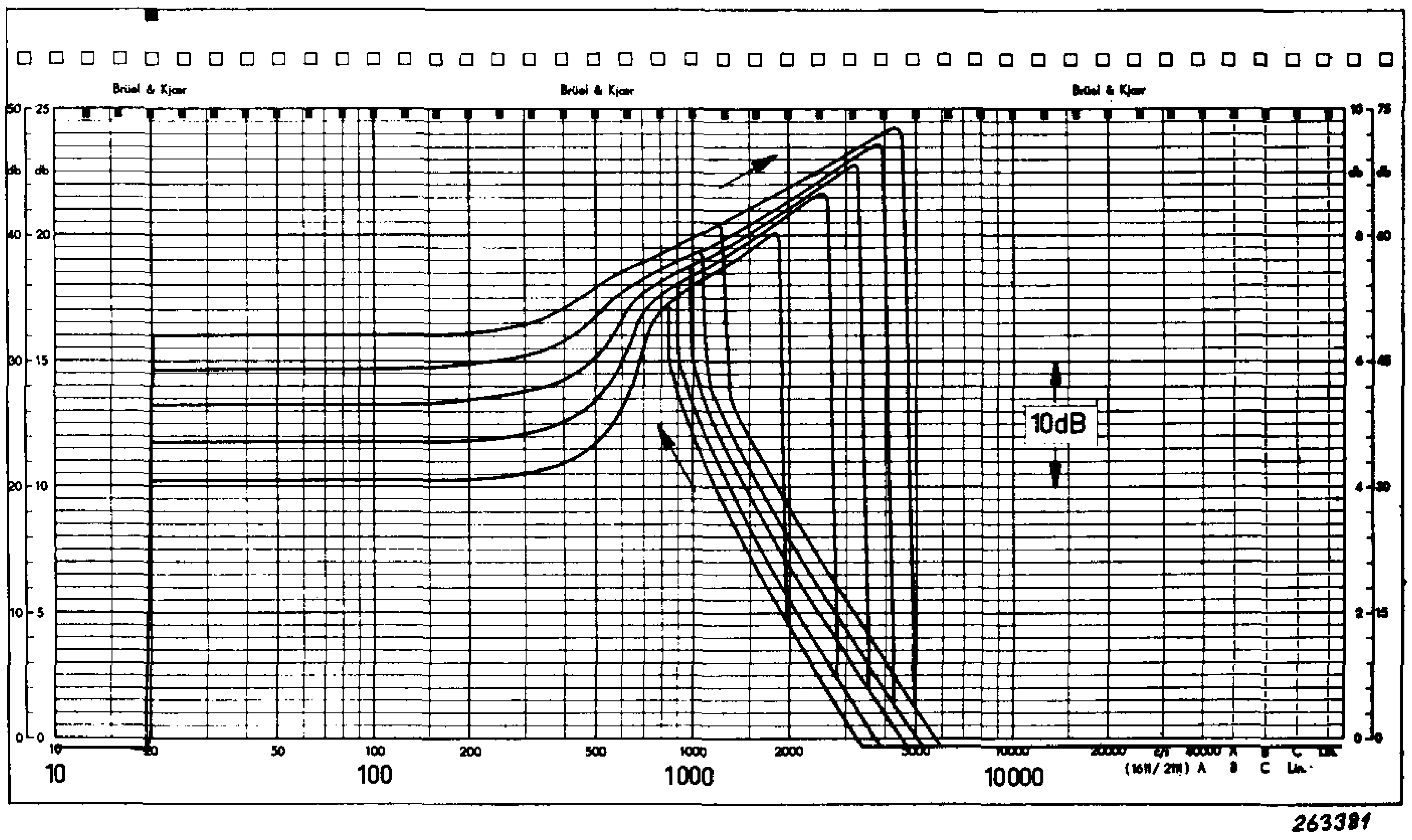


Fig. 9. Frequency response curves for the "hardening" spring-system at various levels of excitation. Curves recorded as the "output" r.m.s. velocity level (voltage) for a constant input velocity (voltage) signal sweeping slowly upwards in frequency, and for reversed direction of the sweep are both shown.

Starting with the frequency response of such a system to a sweeping sinusoidal input signal, it was stated earlier in this article that one of the effects of a "hardening" spring characteristic was to move the resonance upwards in frequency with increasing excitation. Fig. 9 demonstrates this clearly. The curves were here recorded by means of the B & K Level Recorder Type 2305 which, when slightly modified, allow the recording of response curves both "forwards" and "backwards". This is an essential feature in the study of circuits with non-linear "reactive" components, as these circuits produce "jumps" in their response curves, the frequency location of the "jump" being dependent upon the direction of the frequency sweep. The reason for these jumps can be seen from Fig. 10. Theoretically the frequency response curve of a "hardening" spring system will show a "bend" as indicated by means of a dotted line in the figure. In the hatched area of Fig. 10 the system is unstable, and this part of the curve can therefore not easily be measured. It can, however, be proved theoretically that the curve must have the shape shown.

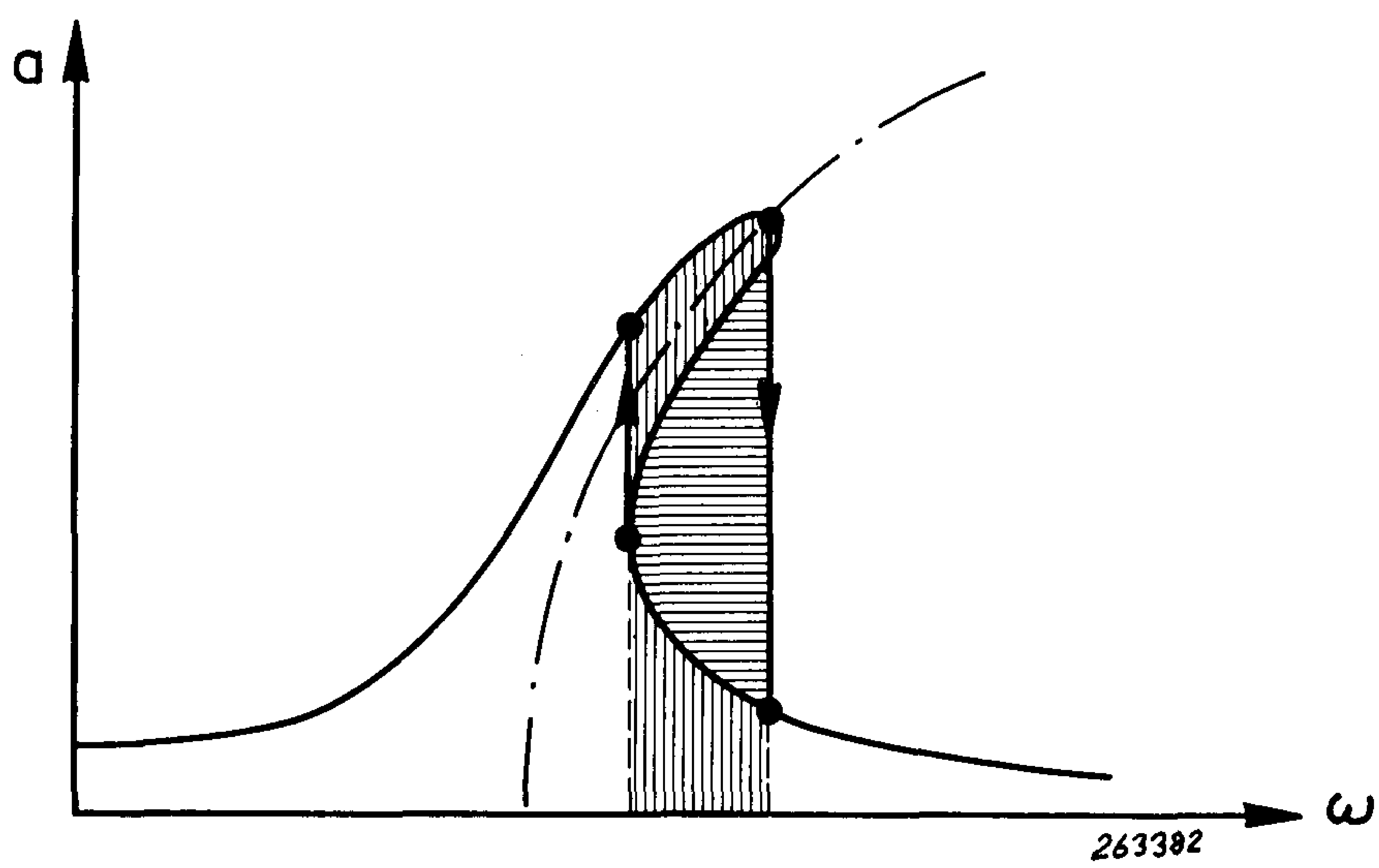


Fig. 10. Theoretical frequency response curve for a "hardening" spring type resonant system.

Now, as soon as the non-linear level of excitation is reached the wave-shape of the output signal from the circuit is distorted, i.e. even if the input signal is a pure sinusoid the output signal will not be sinusoidal. The actual wave shape of the output signal will depend upon whether it is the displacement, the velocity (voltage) or the acceleration of the mass (capacitor) that is being studied. Assuming that the sinusoidal *velocity* of the foundation on which the spring-mass system is mounted is kept independent of frequency and load (constant input voltage, Fig. 9) the acceleration of the mass (current through the capacitor) will have a shape as shown in Fig. 11.
The "peaking" effect of the system is clearly noticed.

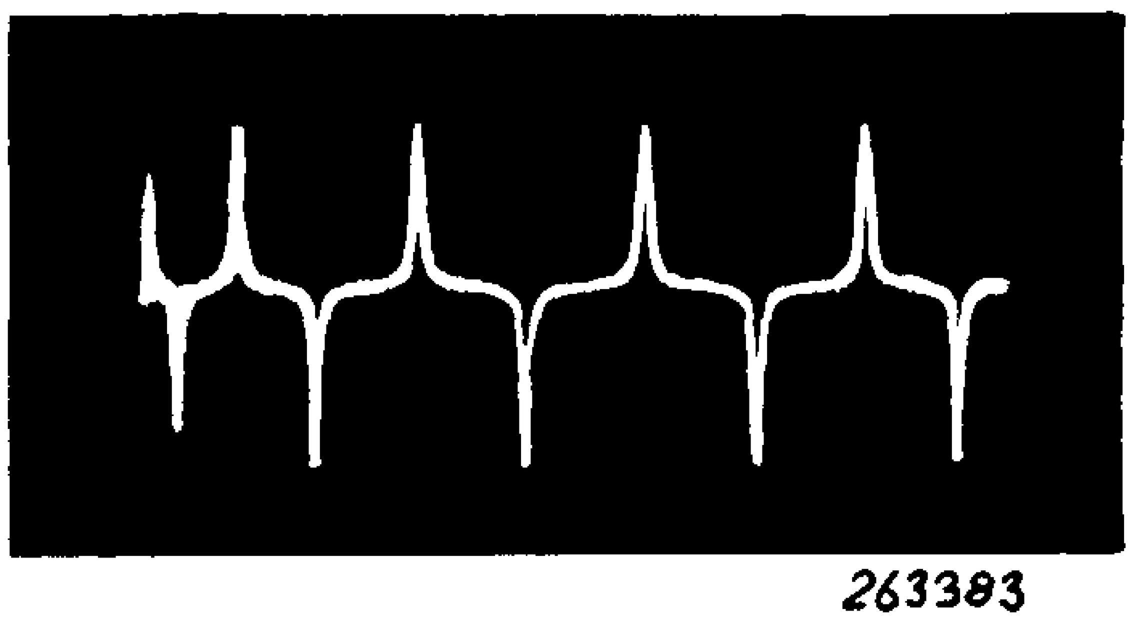


Fig. 11. Typical output acceleration wave-shape photographed off the screen of an oscilloscope. The picture of the wave-shape was taken just before the "jump" during an upwards sweep in frequency.

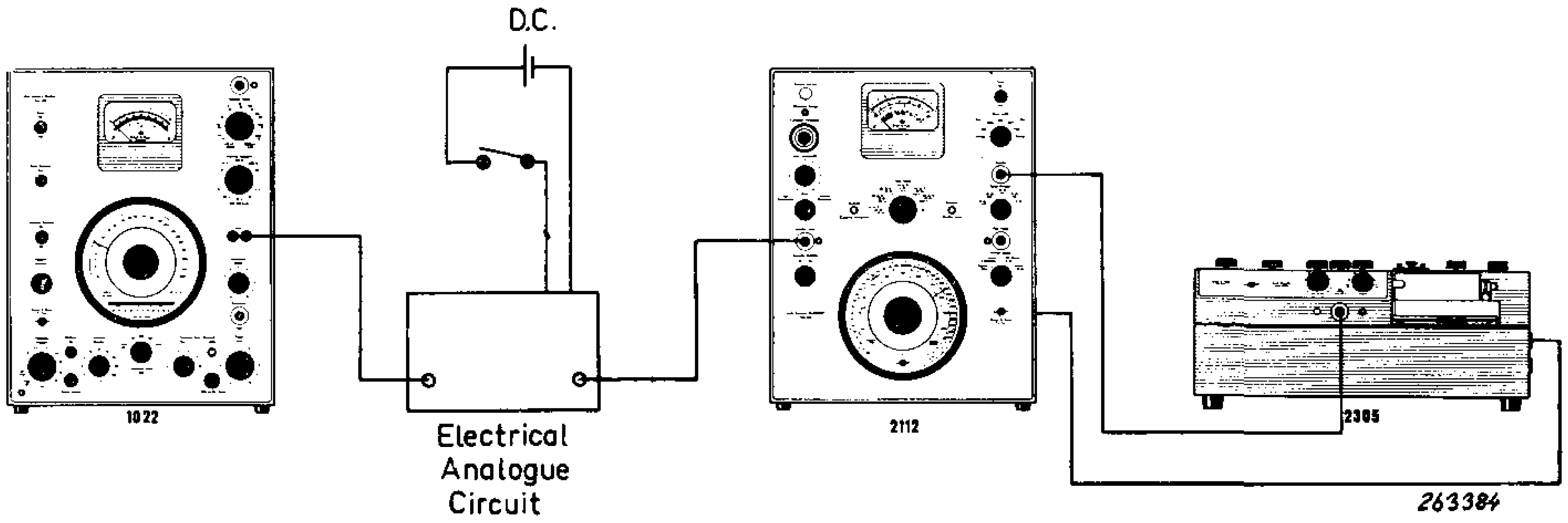


Fig. 12. Measuring arrangement used to record, automatically, the harmonic analysis of the output signal.

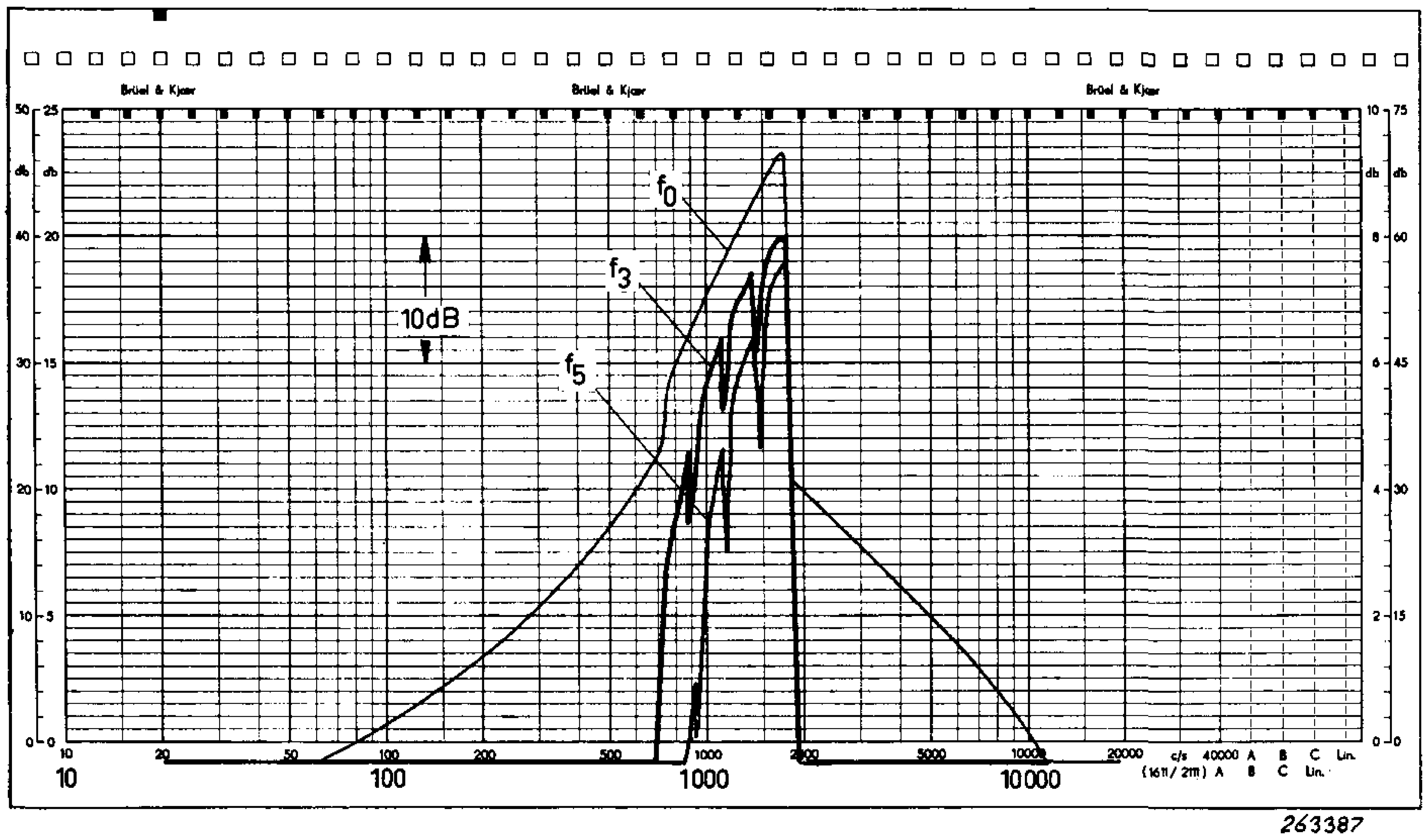


Fig. 13. Resonance curve and the corresponding harmonic components produced by the non-linearity recorded, automatically, by means of the arrangement shown in Fig. 12. The resonance curve was measured as the r.m.s.-value of the output acceleration with constant input velocity level.

Actually, if the system had a clear limiting effect the acceleration would tend towards a sharp peak at each half cycle, as the excitation level was increased. Such peaks produce frequency spectra which do not fall off with the order of harmonic, and the acceleration level of the mass would be equally strong at the *fundamental frequency and all its harmonics.* This is an extremely serious case if the supported system contains several degrees-of-freedom whose resonances coincide with the harmonics of the non-linear resonance.
Physically the peaking effect can be easily visualized in that a "hardening" spring tends to limit the displacement of the mass and thus to increase the transferred force. Also as the distortions do not normally take place suddenly, but increase with increasing vibratory level, an excitation of the system, not only *at* its resonance frequency but also in the *neighbourhood* of this frequency, will cause considerable distortion of the wave shape and the production of harmonics. To demonstrate this a measuring arrangement was set up consisting of a Beat Frequency Oscillator Type 1022, an Audio Frequency Spectrometer Type 2112 and a Level Recorder Type 2305, Fig. 12. It was then possible to automatically record the harmonics of the order of 2, 3, 4 and 5 of a sweeping input signal. Only the third and fifth harmonics are of interest in a symmetrical system, and the result of an automatic recording is seen from Fig. 13 together with the resonance curve for the same excitation level. The small "dips" in the curves correspond to the switching frequencies between the filter bands in the Spectrometer. Finally, a more comprehensive frequency analysis at various levels of distortion was carried out. (Fig. 14).

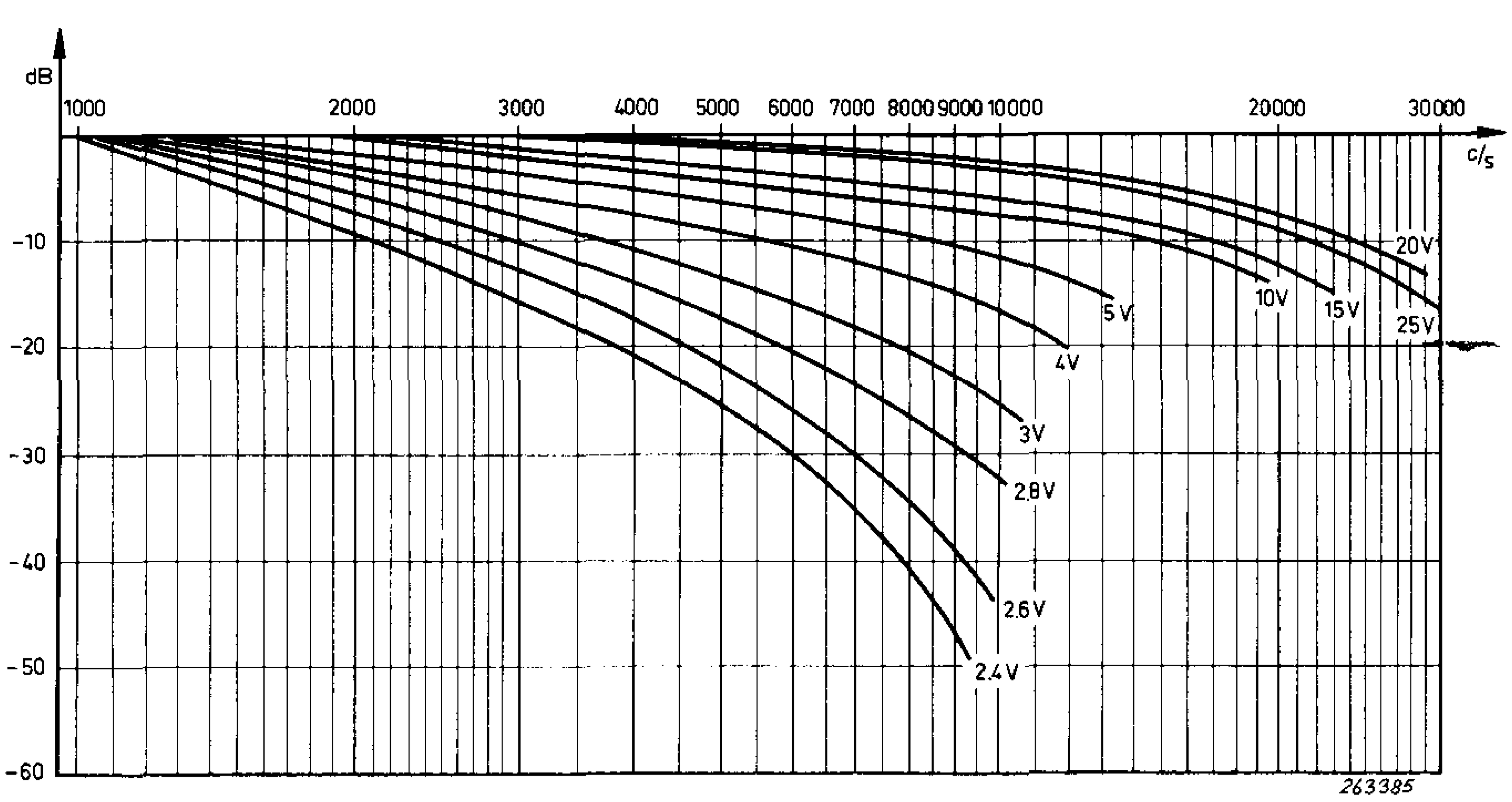


Fig. 14. Results obtained from frequency analysis of the acceleration output at various levels of excitation. The curves are plotted in dB re. the level of the fundamental frequency.

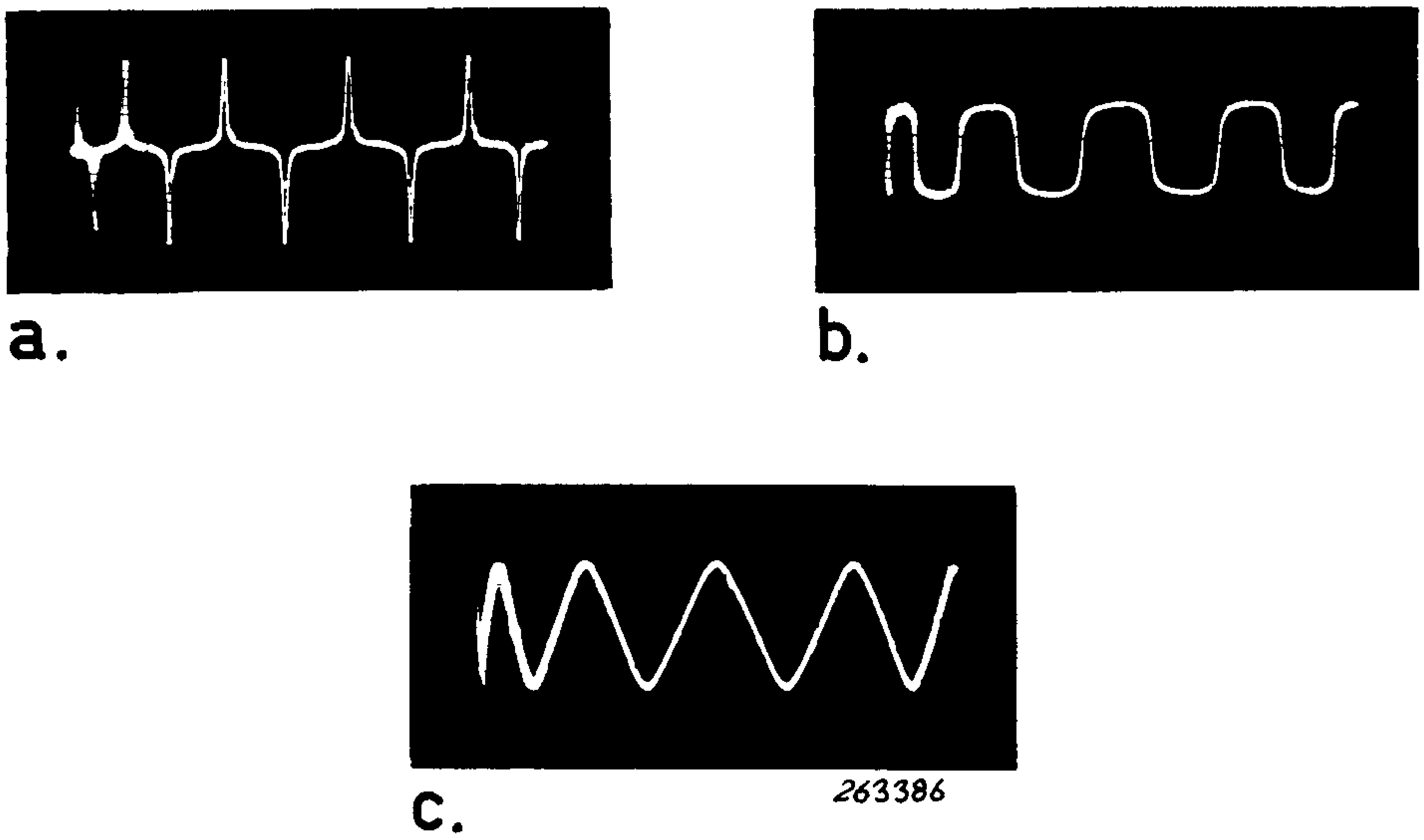
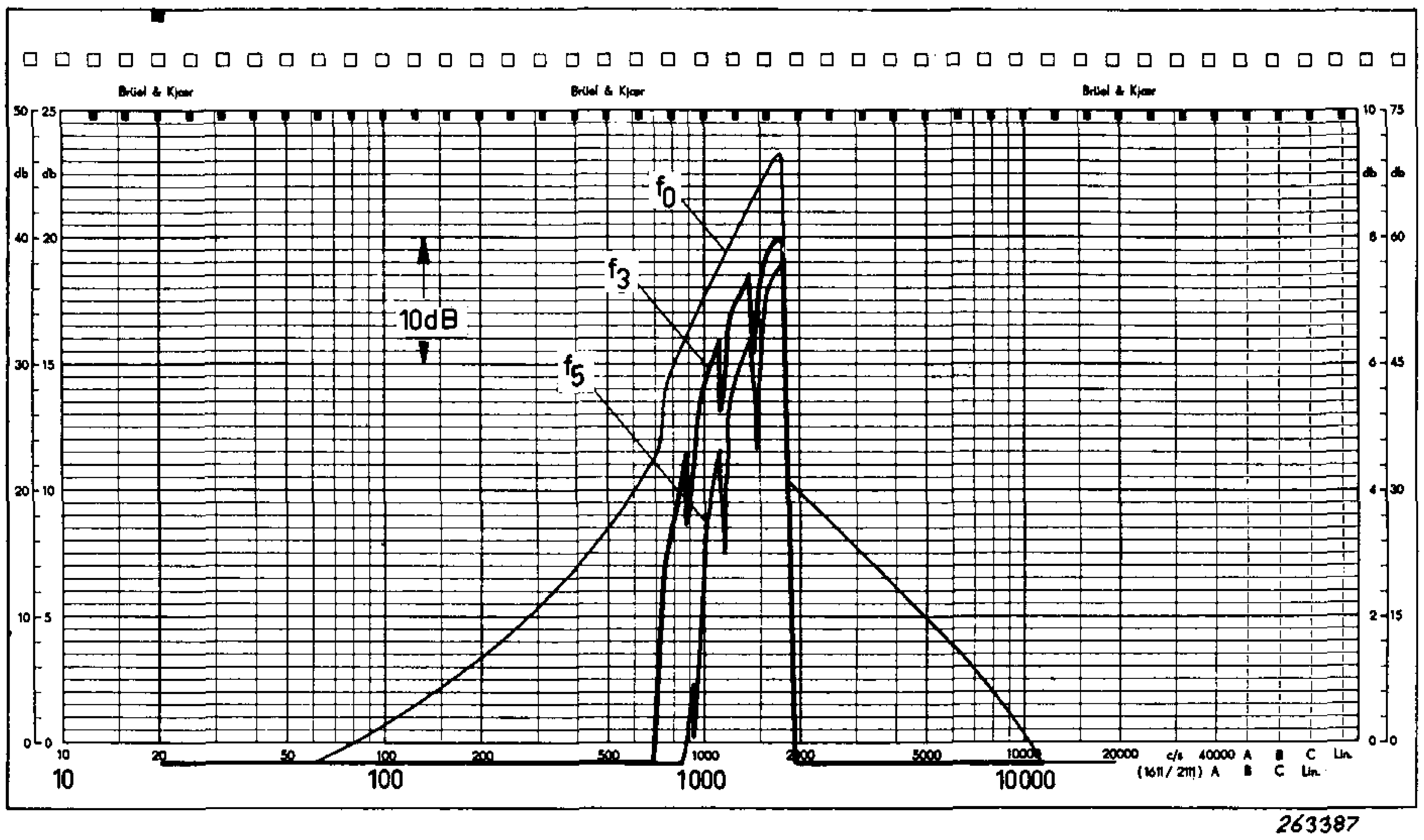
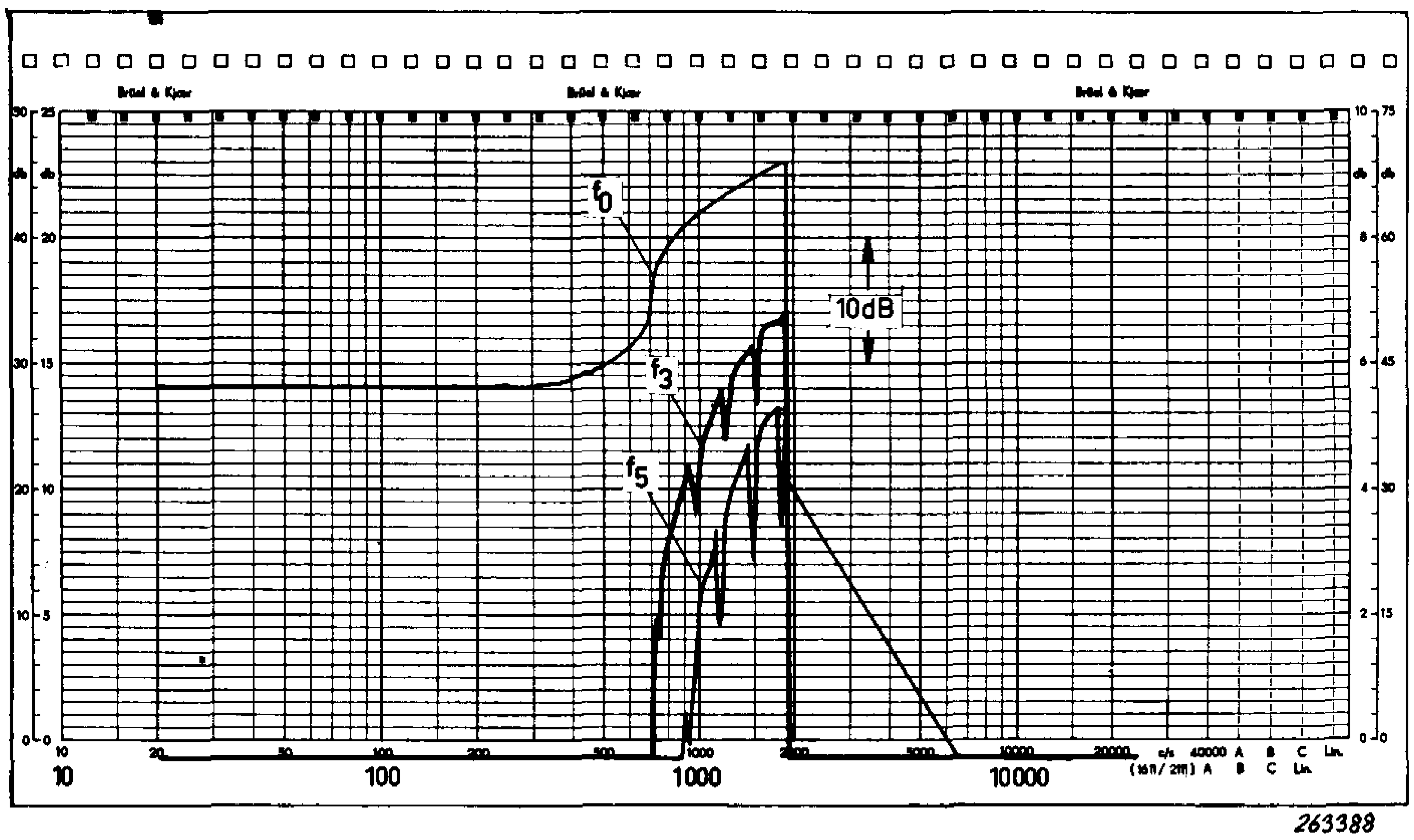


Fig. 15. Typical output wave-shapes corresponding to a fixed level (and frequency) of excitation.
a) The acceleration signal.
b) The velocity signal.
c) The displacement signal.

So far only the characteristics of the acceleration "output" has been considered. However, if the characteristics of this signal are known it is a very simple matter to estimate the characteristics of the velocity and displacement output, as these can be obtained by simple integration. A single integration of the signal shown in Fig. 11 gives a signal of the square-wave type, Fig. 15b, which will then correspond to the velocity of the mass ($v = \int x dt$, where v = velocity, x = displacement and t = time). This signal is obtained for the voltage across the capacitor and can be displayed on the screen of an oscilloscope.
A further integration gives the displacement of the mass, which will have a shape as shown in Fig. 15c.
The frequency spectra of the velocity and displacement signals are related to the spectrum of the acceleration signal in a similar manner. Here the process of integration is equal to a frequency "weighting" of — 6 dB/octave, so that the spectrum of the velocity will "fall off" with frequency at a rate of — 6 dB/octave relative to the acceleration spectrum. Similarly, the frequency spectrum of the displacement signal will "fall off" with frequency at a rate of — 12 dB/octave relative to the acceleration signal. A clear picture of the change in frequency response and spectrum can be obtained from Fig. 16, where automatic recordings of the r.m.s. response (fundamental + harmonics) as well as the third and fifth harmonics are shown for all three cases.



a)



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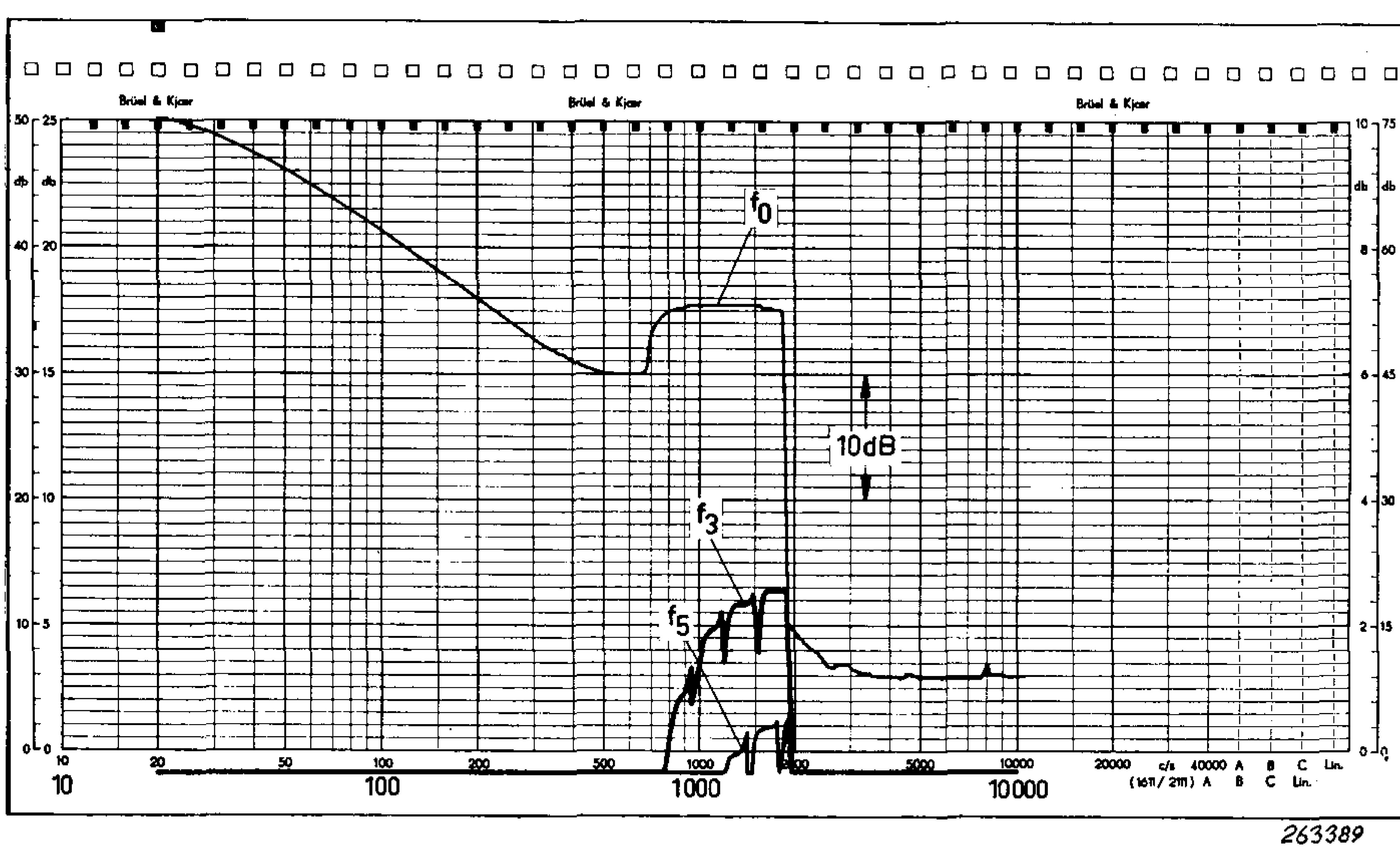


c)

Fig. 16. Recordings of the r.m.s. response as well as the third and fifth harmonic of the output signal vs. frequency with constant input velocity.
a) The output acceleration.
b) The output velocity.
c) The output displacement.

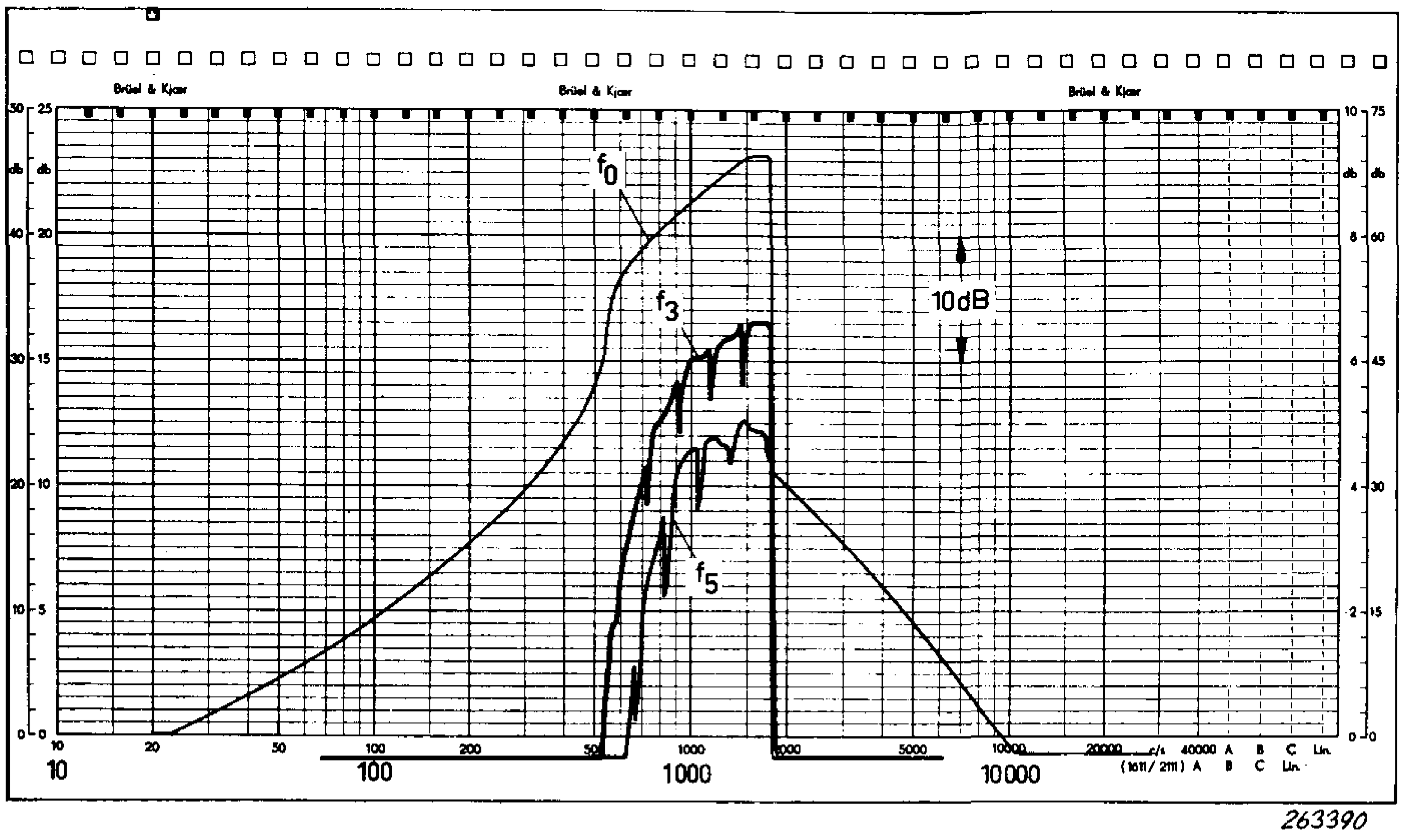


Fig. 17. Output velocity (r.m.s.) response and harmonic analysis vs. frequency with the system excited by constant displacement input.

The frequency response curves, wave-forms and spectra shown up to this point have been measured with constant velocity (constant voltage) excitation of the system. If, on the other hand, the *input* consists of a constant displacement signal, how will the response change? This can be readily seen from the well known relationship between the displacement, x, and the velocity, v: $x = \int v dt$. To keep x constant v must vary in such a way that $\int v dt =$ constant. When v is sinusoidal then $\int v dt = \int v_o \sin(\omega t)\, dt = -\frac{v_o}{\omega} \cos(\omega t)$.
To keep this expression constant with frequency v must increase with ω at a rate of 6 dB/octave. In the mobility analogy this corresponds to an increase in input voltage of + 6 dB/octave. With sinusoidal excitation it can, furthermore, be seen that the *wave-shape* of the input signal does not change even though the amplitude changes. It is therefore to be expected that the *type* of wave-shape, and thus the *type* of frequency spectrum, of the output signal from the system will remain the same, whether the input consists of a constant acceleration, velocity or displacement signal. However, the variation in output spectrum with frequency of the input signal will be different from that shown in Fig. 16, and so will the actual frequency response curve, Fig. 17.
If it is desired to study the response of the system to a constant acceleration input, this leads to an analogue input voltage which ,decreases 6 dB/octave. Again the *type* of output signal will be the same as before, but the spectrum variation with input signal frequency, and the response curve, will differ from previous results, see Fig. 18. From the preceding discussion and the

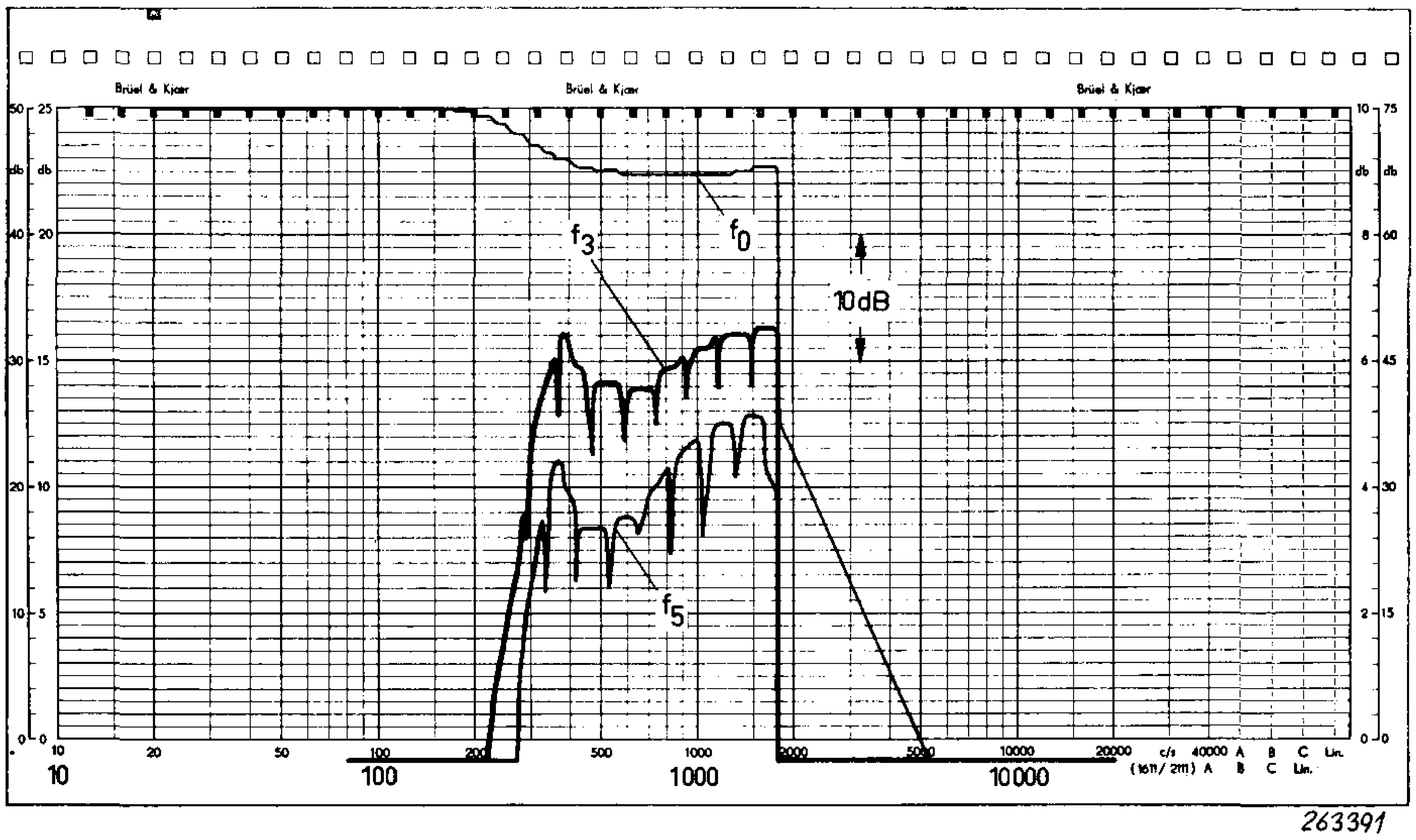


Fig. 18. Output velocity (r.m.s.) response and harmonic analysis vs. frequency with the system excited by constant acceleration input.

curves shown in Figs. 17 and 18 it is clear that, in contrast to linear systems, the shape of the response curve will differ whether the differentiation (or integration) process takes place before or after the signal is applied to the non-linear system. This should be borne in mind when analogue studies are made on such systems, especially if a more complicated input than the pure sine-wave is employed.

b) The "softening" spring system.

Measurements similar to those carried out for the "hardening" spring system have also been performed on a "softening" spring analogue. To minimize the iron-core losses the measurements had to be performed at considerably lower frequency than in the case of the "hardening" spring circuit. This, however, results in a lower Q-value of the resonant system which can be seen from the set of frequency response curves Fig. 19. Here the difference in frequency location of the "jumps" when sweeping forwards and backwards is very small, indicating that the circuit Q is small (Note: The difference in frequency location of the "jumps" depends *both* on the non-linear characteristic of the system *and* on the circuit Q-value).

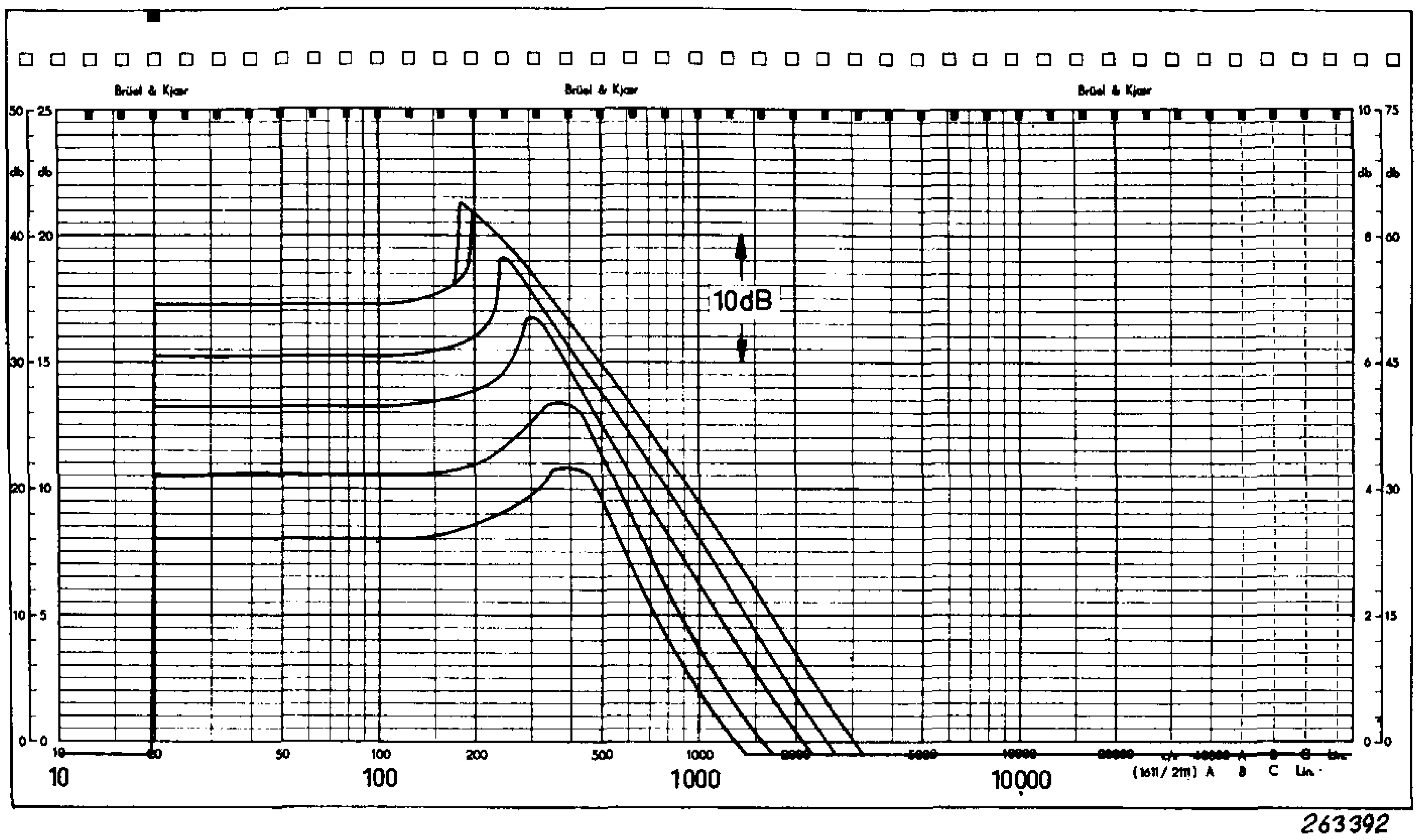


Fig. 19. Frequency response curves at various levels of excitation obtained from measurements on a "softening" spring system.

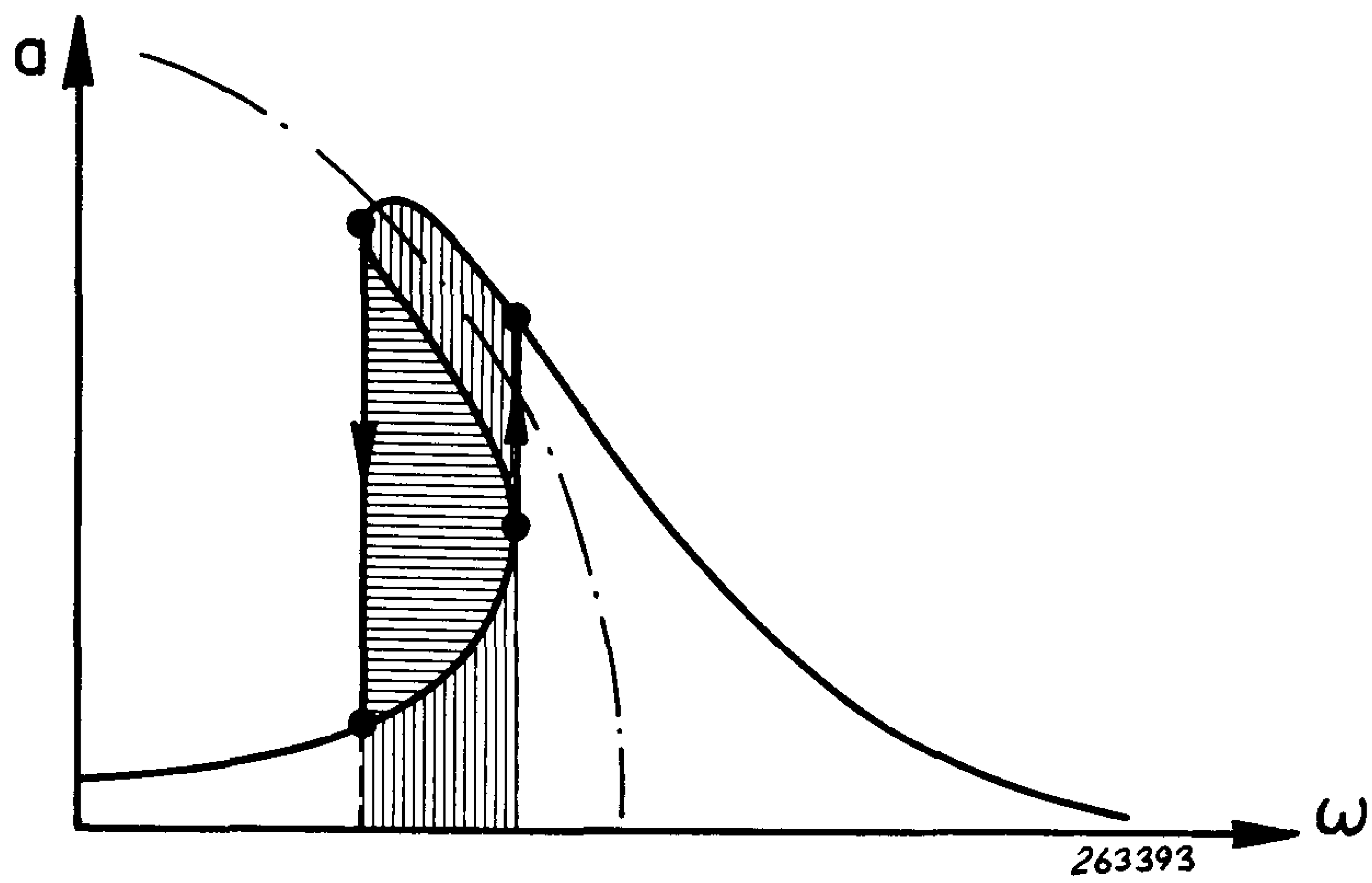


Fig. 20. Theoretical frequency response curve for a "softening" spring type resonant system.

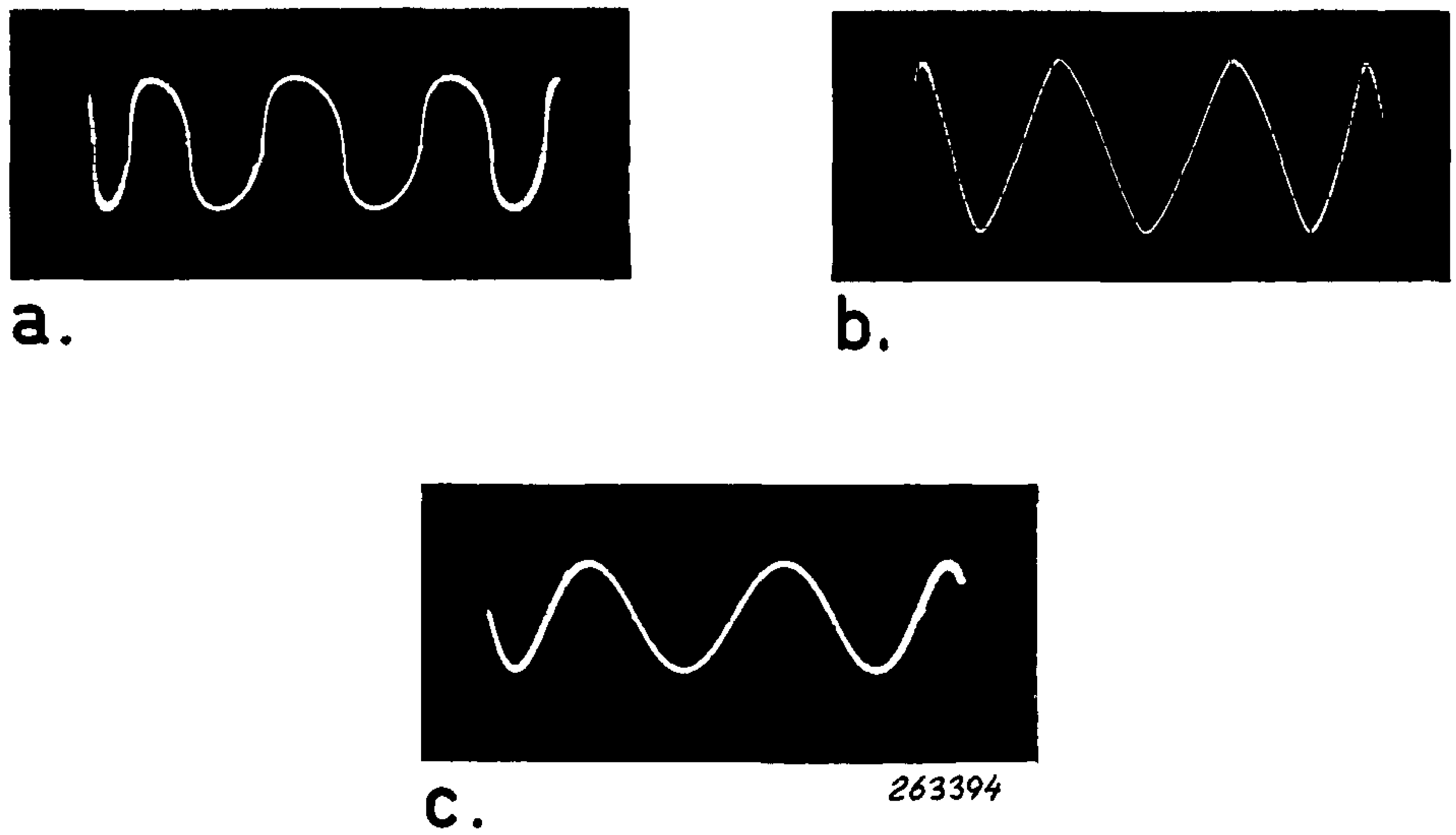
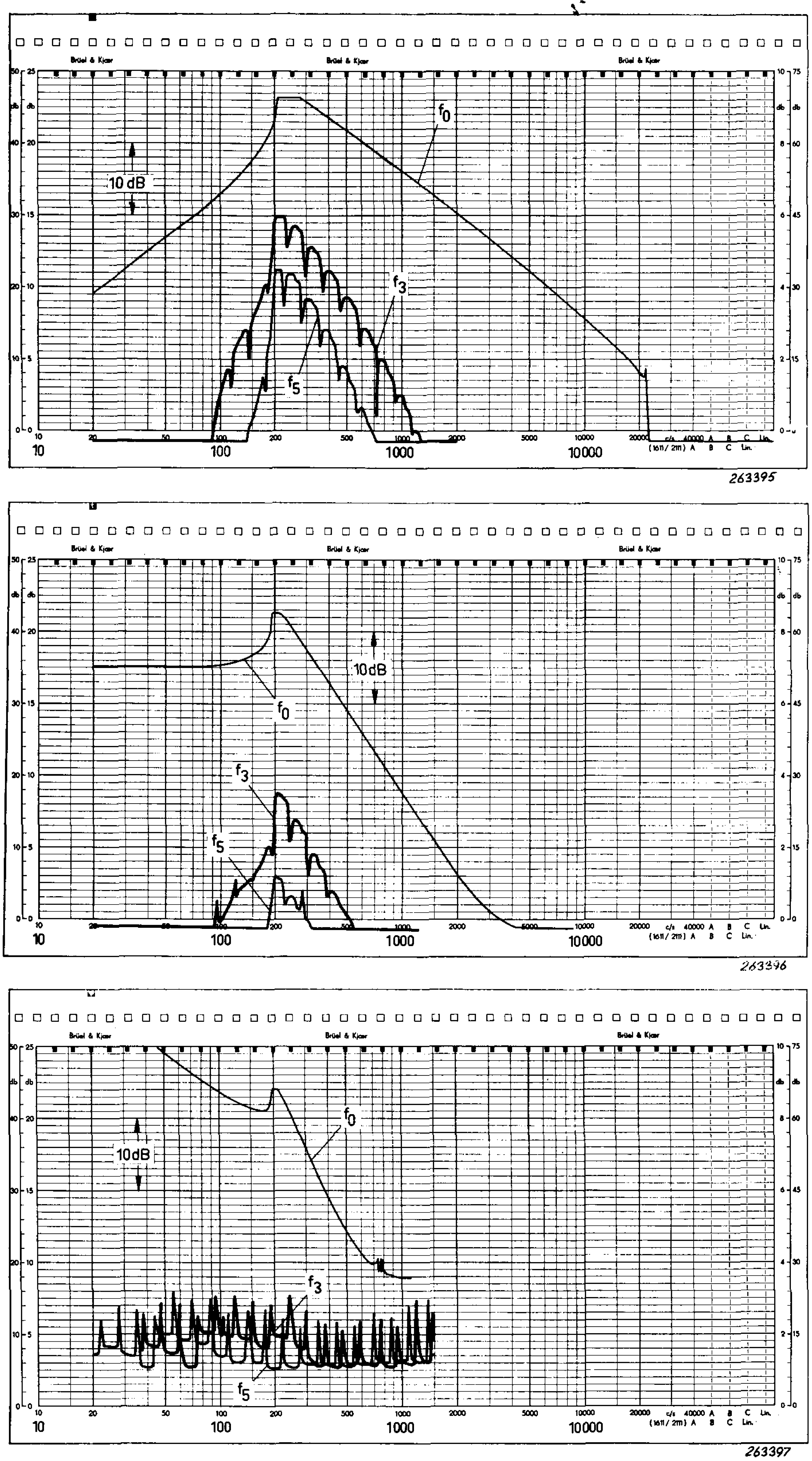


Fig. 21. Typical output wave-shapes at the highest response level.
a) The acceleration signal.
b) The velocity signal.
c) The displacement signal.

Theoretically, the shape of the resonance curve should be as shown in Fig. 20. The response curves given in Fig. 19 were obtained from measurements with constant input velocity (voltage), and Fig. 21 shows the output wave-shapes, just after the jumps, i.e., at the highest excitation level. From the figure it is seen that also in this case the acceleration of the mass (capacitor) contains a fair amount of harmonics. However, this signal can, in the "worst" case become a square wave which contains harmonics that drop off with frequency at a rate of 6 dB/octave. Again the velocity signal (voltage across the capacitor) contains a smaller amount of harmonics and the displacement signal is almost a "pure" sine-wave. This is also clear from the automatic recordings shown in Fig. 22.

It is thus seen that for the systems considered here the "hardening" spring system is much more dangerous than the "softening" spring system with regard to the production of serious harmonics.

Finally, Fig. 23 shows some recordings of the velocity of the mass (voltage across the capacitor) when the excitation consists of a constant displacement and constant acceleration signal.

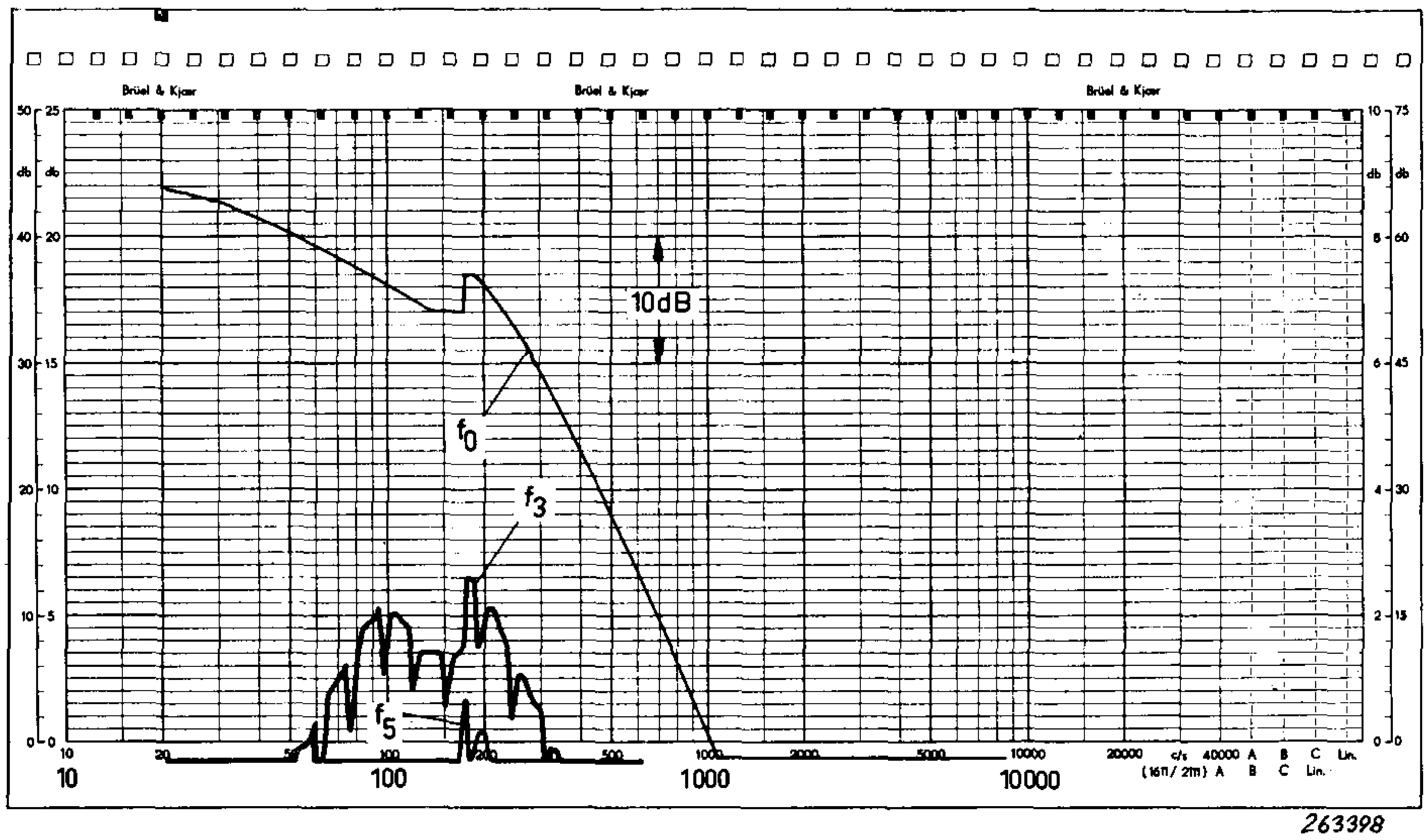


a)

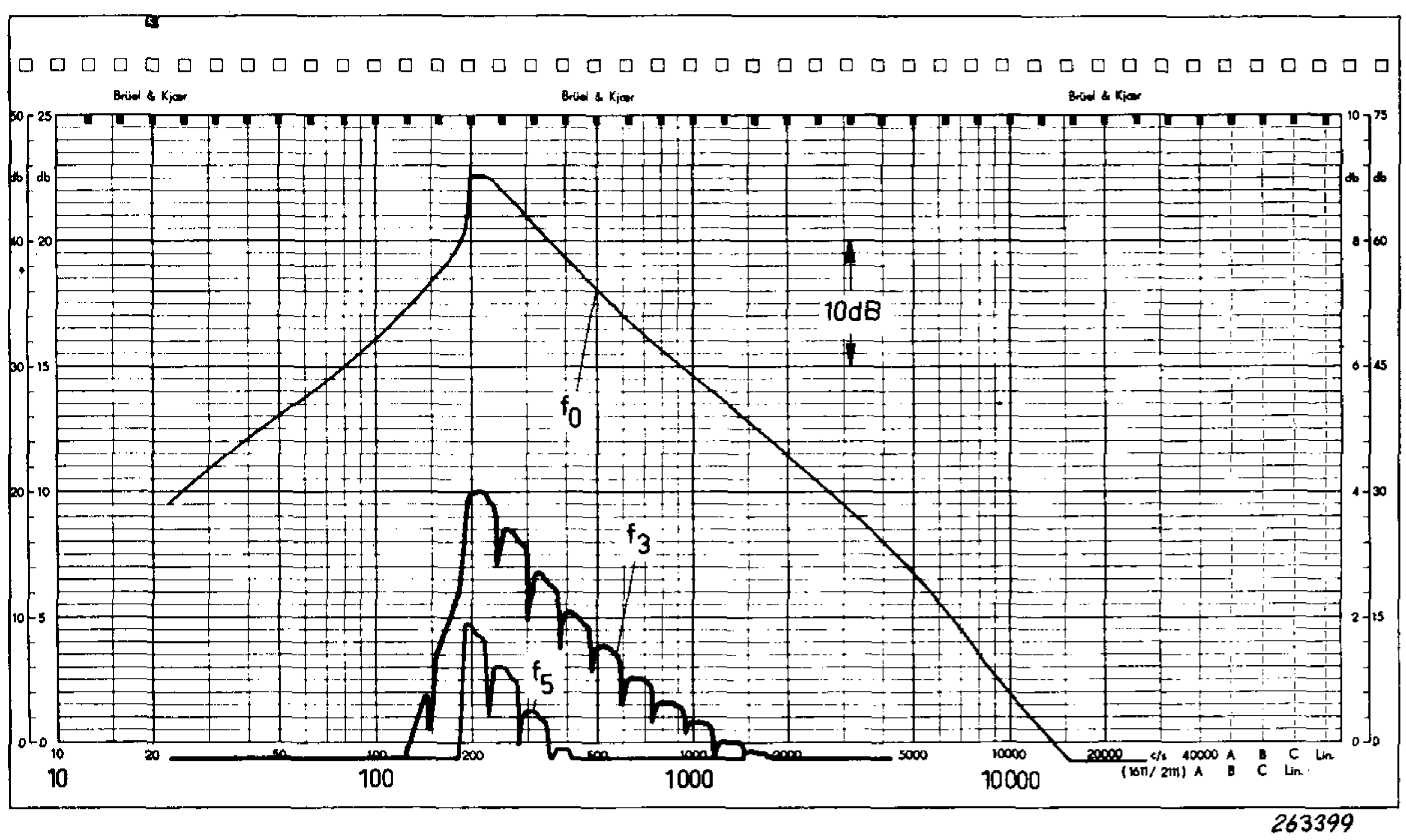
b)

c)

Fig. 22. Recordings of the r.m.s. response as well as the third and fifth harmonic of the output signal vs. frequency with constant input velocity.
a) The output acceleration.
b) The output velocity.
c) The output displacement.



a)



b)

Fig. 23. Output velocity (r.m.s.) response and harmonic analysis vs. frequency with
a) constant displacement input.
b) constant acceleration input.

c) Systems with velocity dependent damping.

Non-linear velocity-dependent damping can be of two kinds, in one case the damping increases with increased velocity level, while in the second case the damping decreases with increased velocity level.
A non-linear increase in damping with increased excitation may be relatively common in practice, but if the non-linearity is small, its effect will be negligible. On the other hand, if the non-linearity is great a "dangerous"

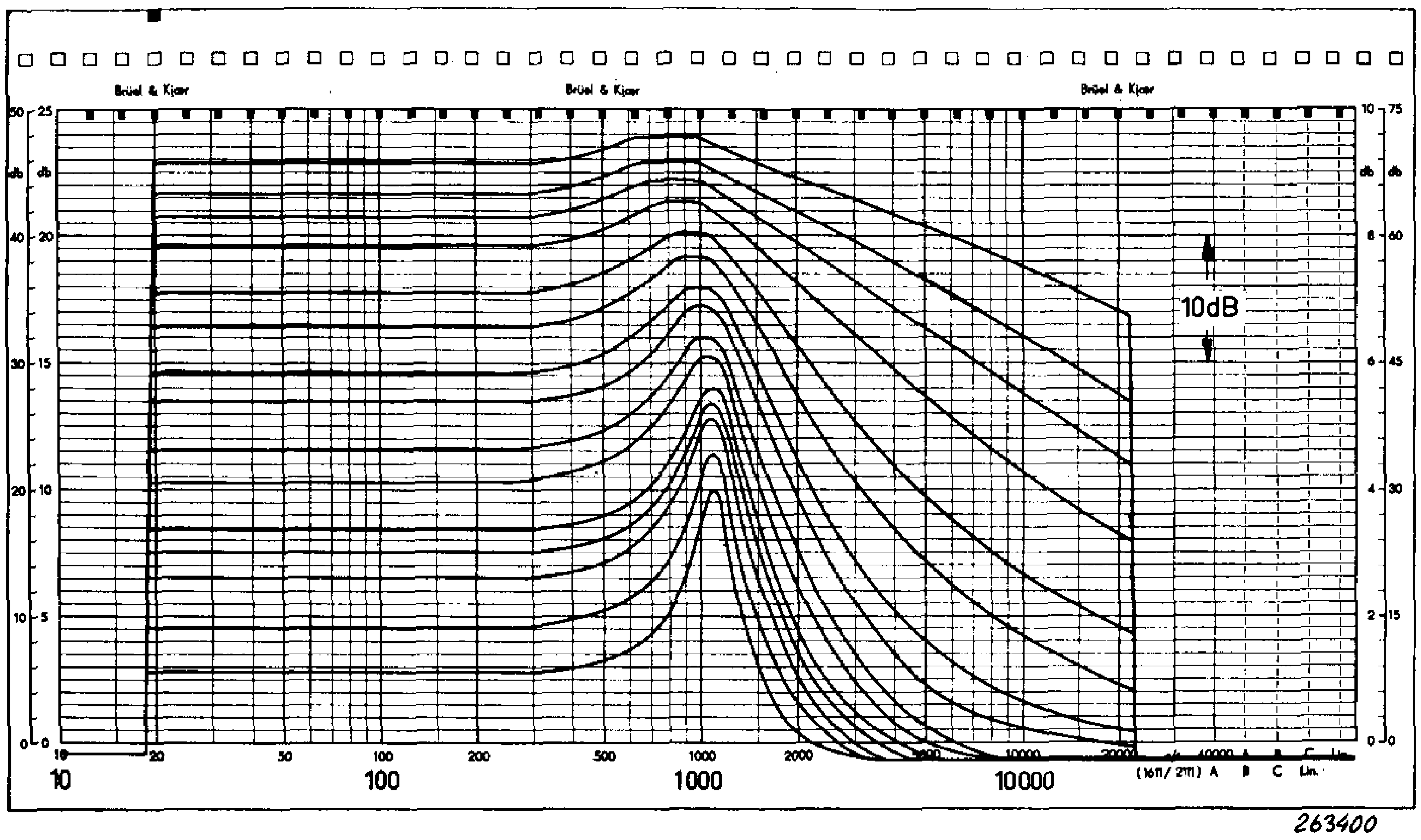


Fig. 24. Frequency response curves (output r.m.s. velocity) obtained for various levels of excitation constant velocity input.

situation can be developed as will be illustrated in the following. Using the analogue circuit given in Fig. 6a frequency response curves as shown in Fig. 24 can be recorded. These curves clearly illustrate how the increase in damping tends to "eliminate" the resonance build-up*). Actually, the system gradually degenerates into a low-pass "filter" with a high frequency drop-off of 6 dB/octave (R.C. circuit). However, as the excitation increases the damping increases much more rapidly (R in the analogue circuit becomes smaller) and therefore the resonance does not only degenerate into a low-pass "filter" but the cut-off frequency of the "filter" also increases with excitation.

The situation may thus be reached where the vibration of the base is transferred directly to the mass over a very wide frequency range above the "original" resonance. This, of course, is an extreme case.

Fig. 25a shows the wave-shape of the velocity of the mass at the highest level of excitation, and the trend towards a triangular wave-shape is noticed. In Fig. 25b the result of an automatic harmonic analysis of the velocity of the mass is shown.

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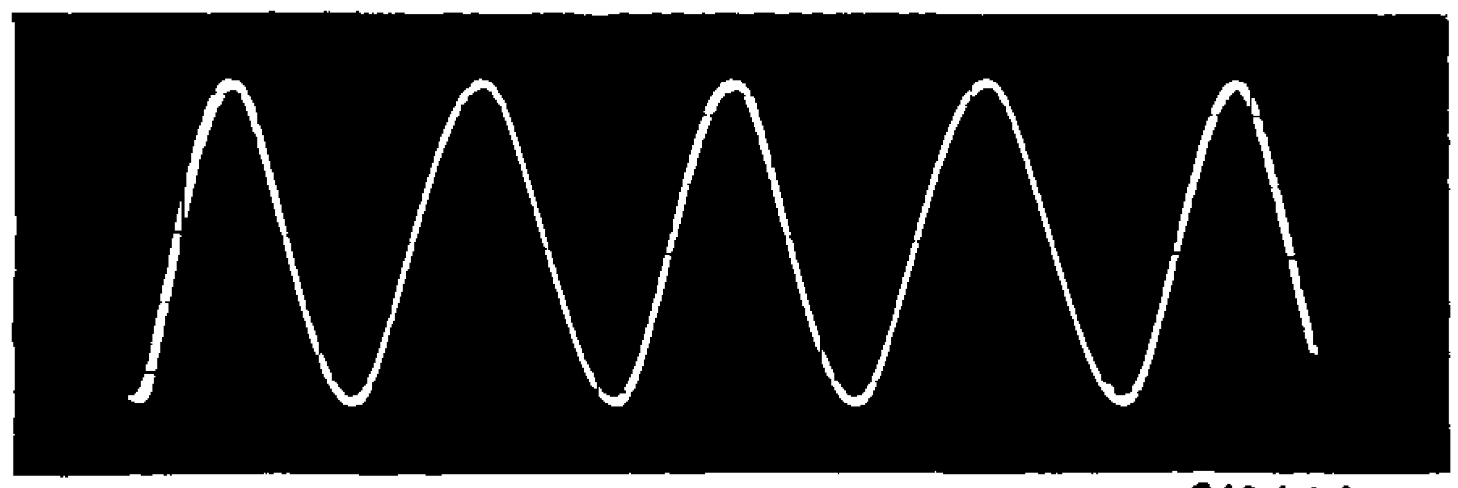
*) The slight decrease in resonance frequency with excitation level noticeable in Fig. 24 is due to a non-linear "mass-action" (non-linear capacity) of the damping element (resistor).



a)

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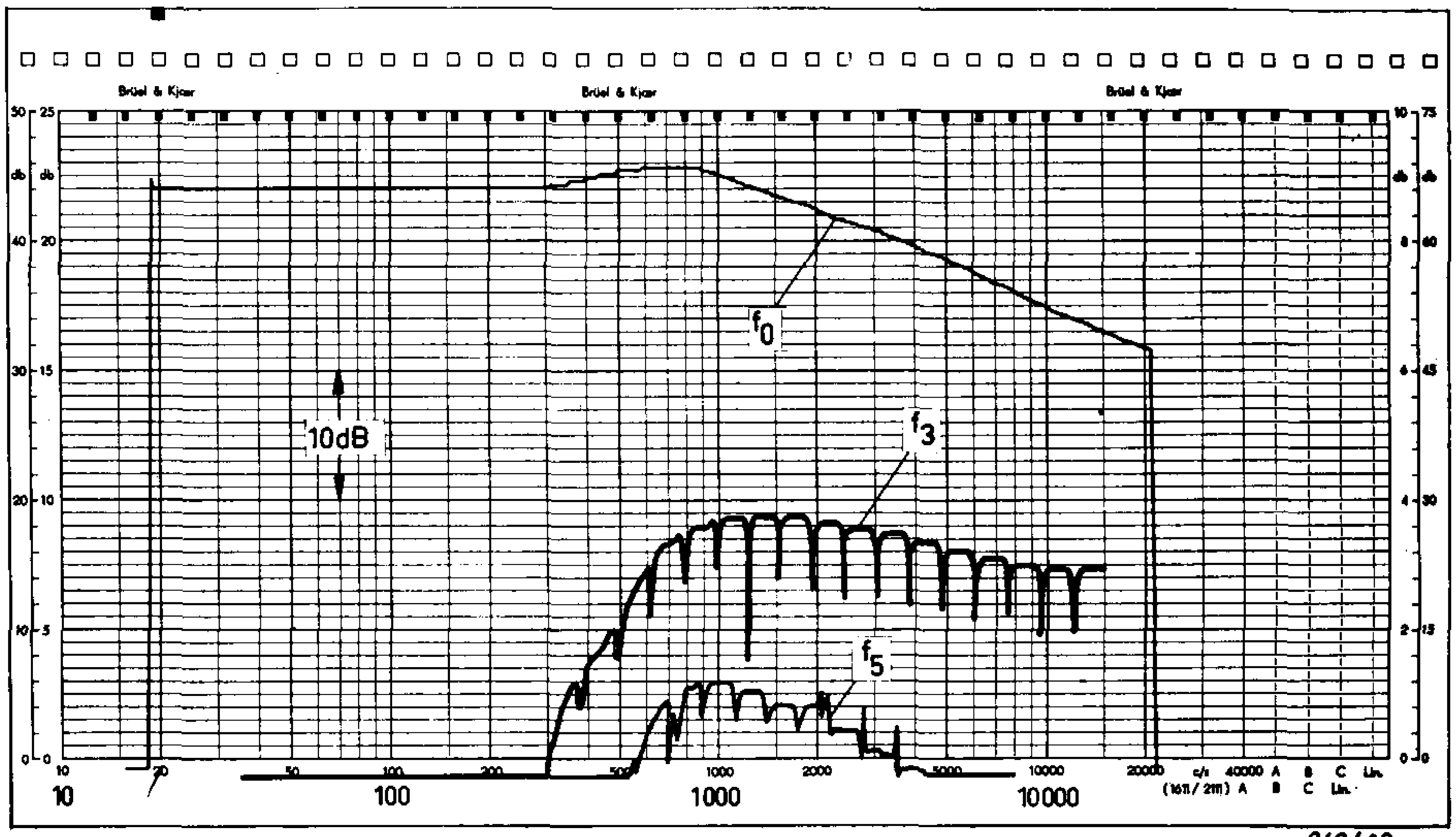
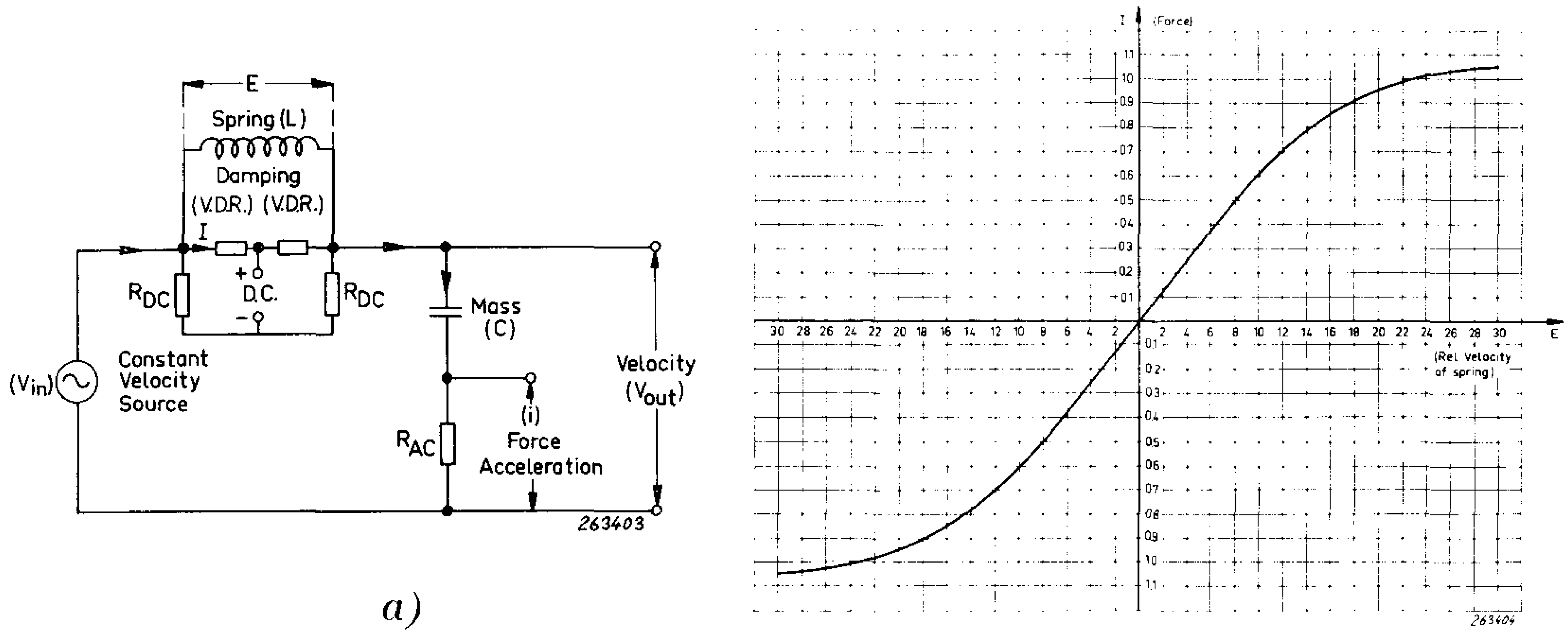


b)

Fig. 25. Typical output wave-shape and spectrum at the highest level of excitation
a) Wave-shape of the velocity output.
b) Frequency spectrum (and r.m.s. response curve) of the velocity output signal vs. frequency.

For the sake of completeness also the case where the damping decreases with increased excitation has been investigated. This case may not likely be found in practice, but it is interesting to compare the various cases of non-linearity with respect to their effects on potential vibration damage. It should, however, be noted that this article deals only with "positive" damping, where self-sustained vibrations do not occur.

An electrical analogue to non-linear damping which decreases with increased excitation level can be produced by V.D.R's in a way similar to the production of a "softening" spring characteristic by means of inductors. Fig. 26a shows the electrical circuit and how the biasing of the V.D.R's can be made. Depending on the direction of the voltage across the inductance, L, the resistance of one of the V.D.R's will increase (due to the reduction in current through it) while the resistance in the other V.D.R. will decrease. Due to the non-linear resistance of the V.D.R.'s the net result will be an overall increase in resistance, and thus a decrease in damping. The operating characteristic of the V.D.R.-circuit is shown in Fig. 26b.



a)

b)

Fig. 26. Electrical circuit and operating characteristic for the V.D.R's used to produce a "velocity" dependent damping which decreases with the level of excitation.

a) Electrical analogue circuit.

b) Typical operating characteristic for the biased V.D.R.-combination.

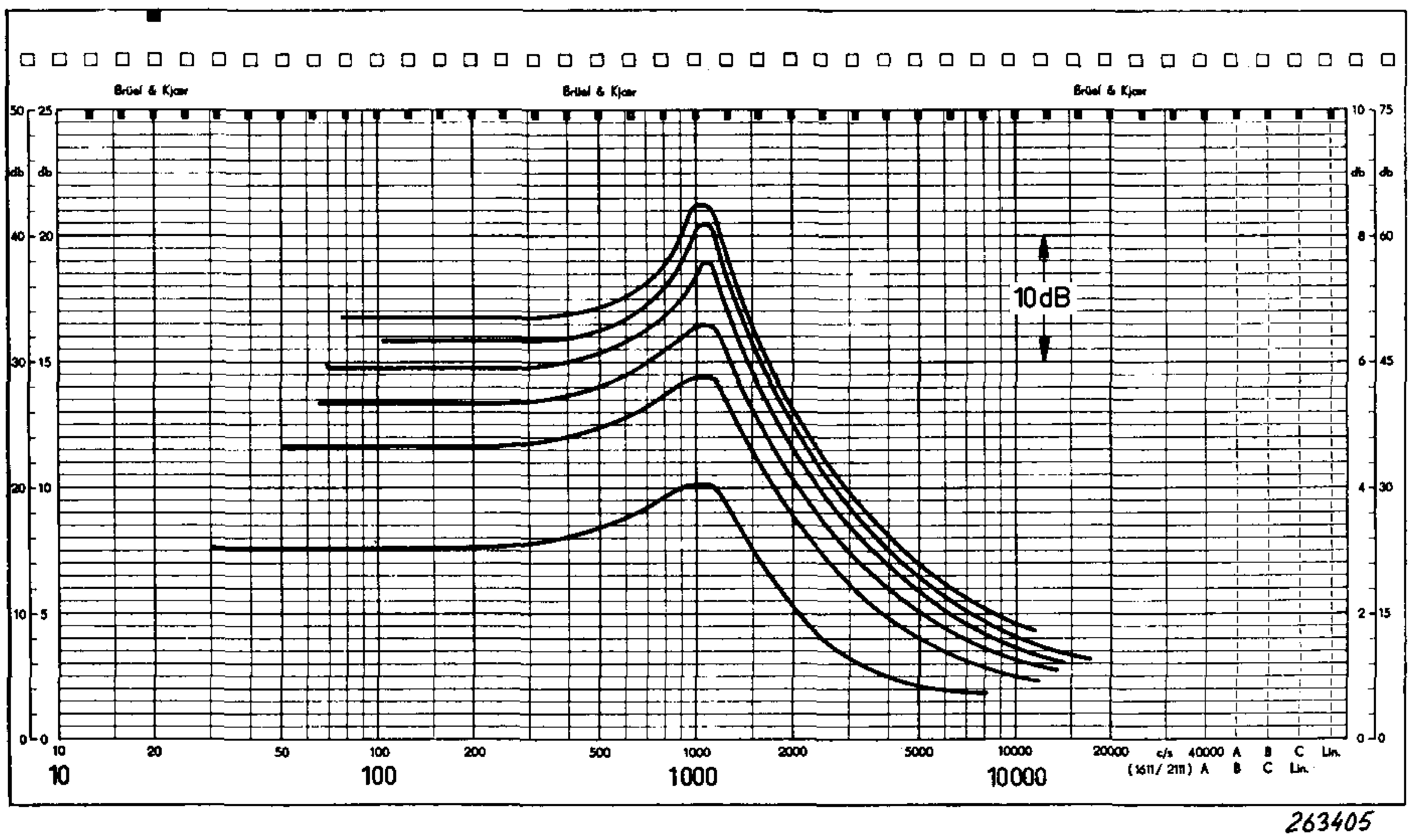


Fig. 27. Frequency response curves (output r.m.s. velocity) recorded for various levels of excitation (constant velocity input).

Fig. 27 shows some typical frequency response characteristics of the velocity of the mass (voltage across the capacitor), and the non-linearity in the resonance build-up is clearly noticed. Also, the higher the Q-value of the system becomes, the "sharper" is the high frequency cut-off and the better the "filtering" effect. It is therefore to be expected that the harmonic content of the wave will be fairly small. An automatic harmonic analysis readily verifies this, as can be seen from Fig. 28, where the third and fifth harmonics are recorded on the same chart paper as the response curve. The measurements were made with an excitation corresponding to the highest vibration level shown in Fig. 27.

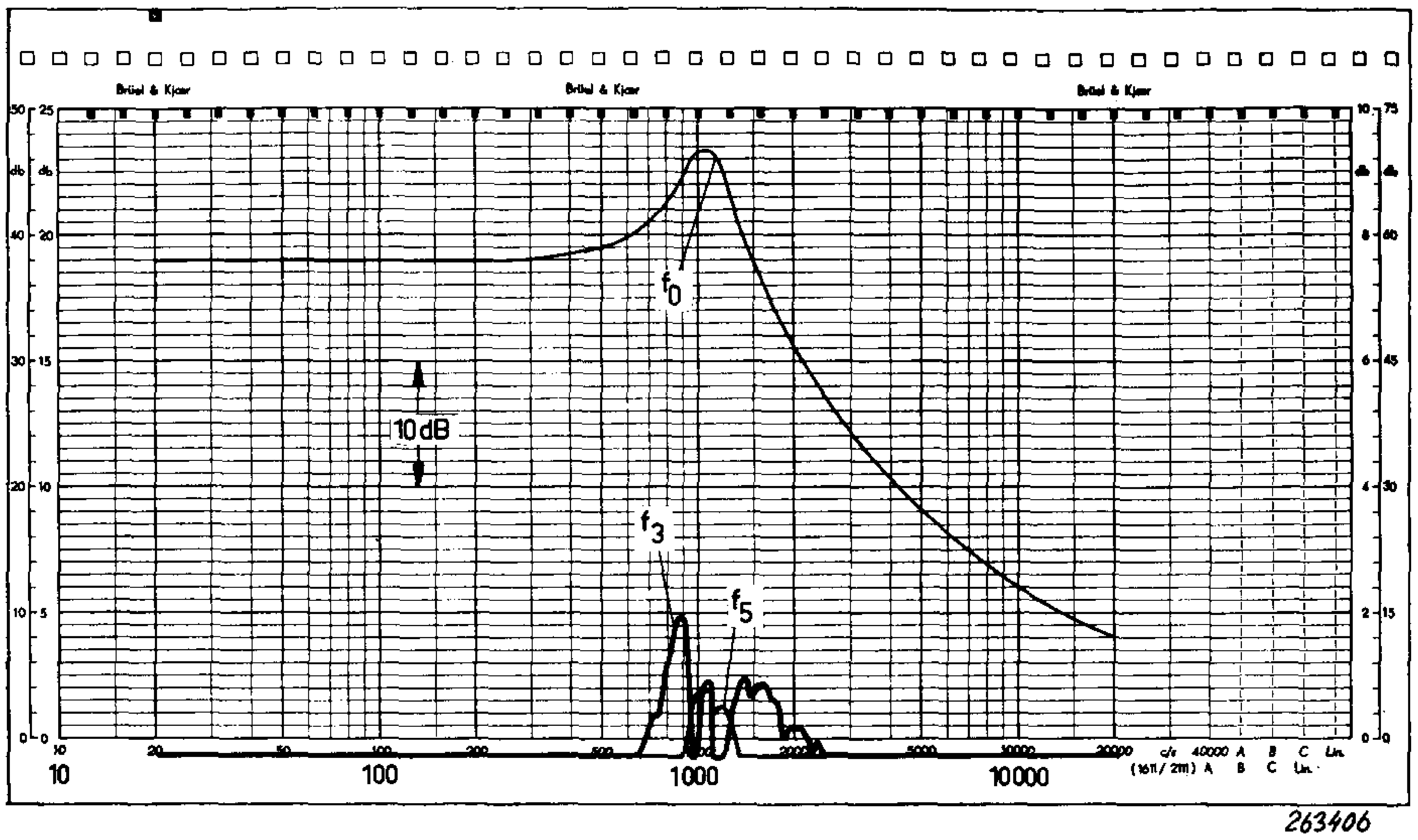


Fig. 28. Automatic harmonic analysis of the output velocity signal together with the corresponding frequency response curve.

Two Degrees-of-Freedom System.

The electrical analogue used in the studies described in the following is shown in Fig. 7b. As mentioned earlier only the "first" resonance is non-linear, while the "second" resonance is linear and separated from the "first", by means of an electronic amplifier, to avoid coupling between the two systems.

The cases which will be of greatest interest are when the "second" resonance is tuned to one of the harmonics produced by the "first" (non-linear) circuit.

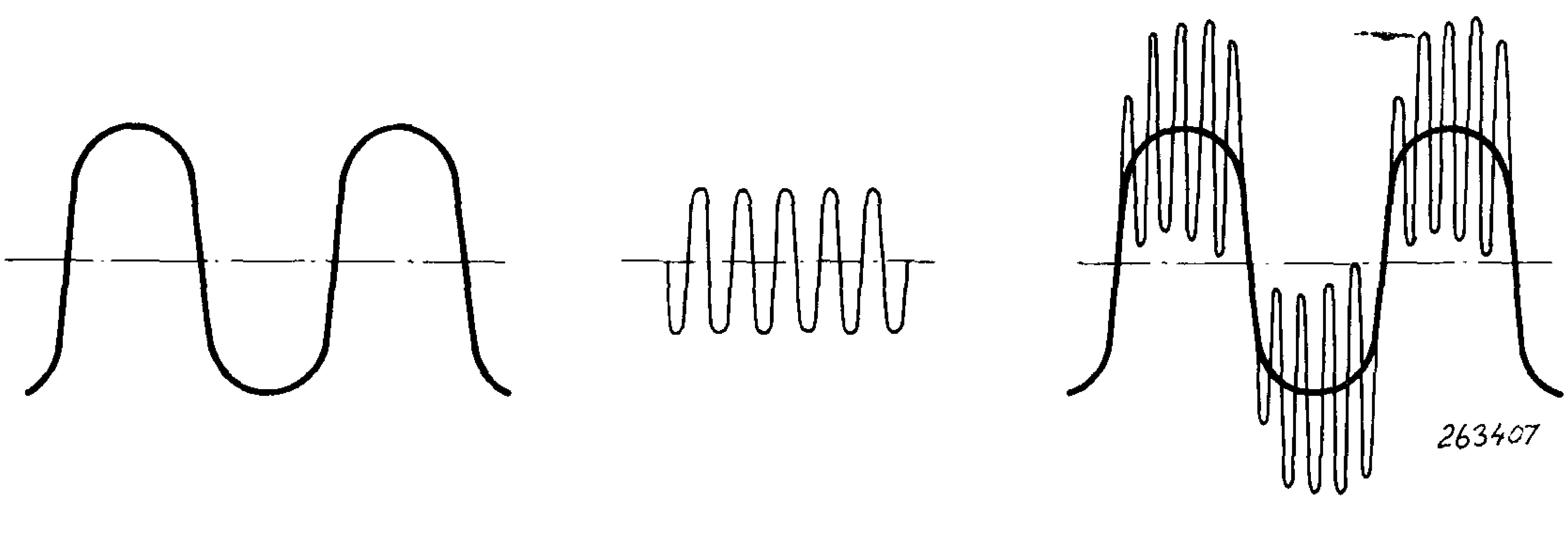


Fig. 29. Sketch showing the "superposition" of the output from the first and the second resonance when the second resonance has a very high Q-value.

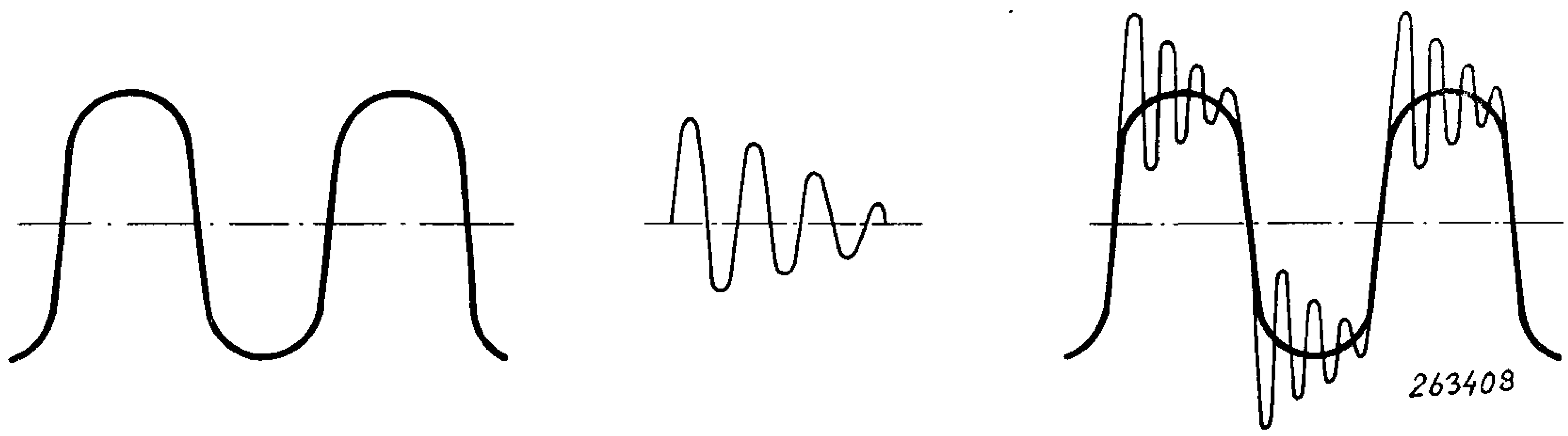
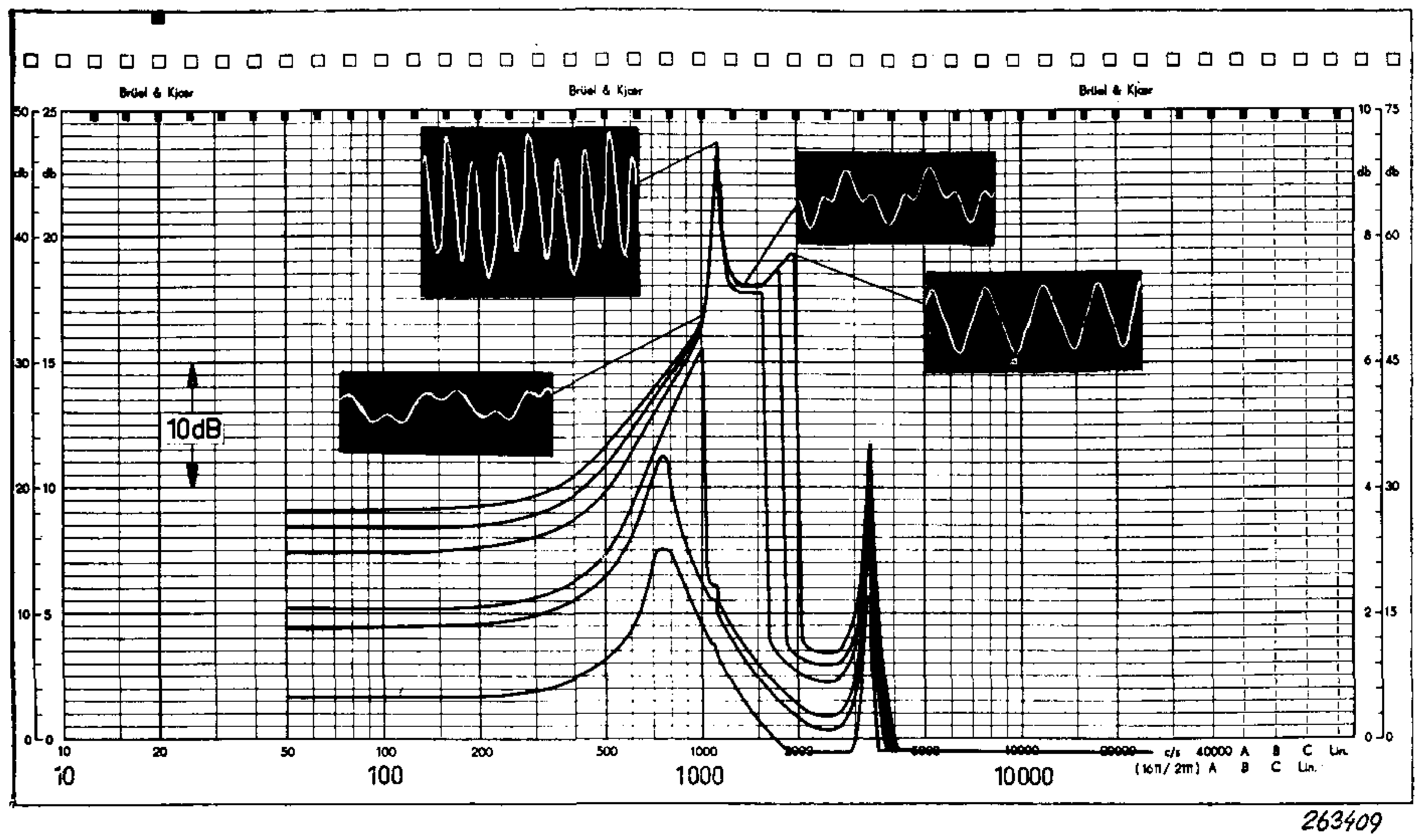


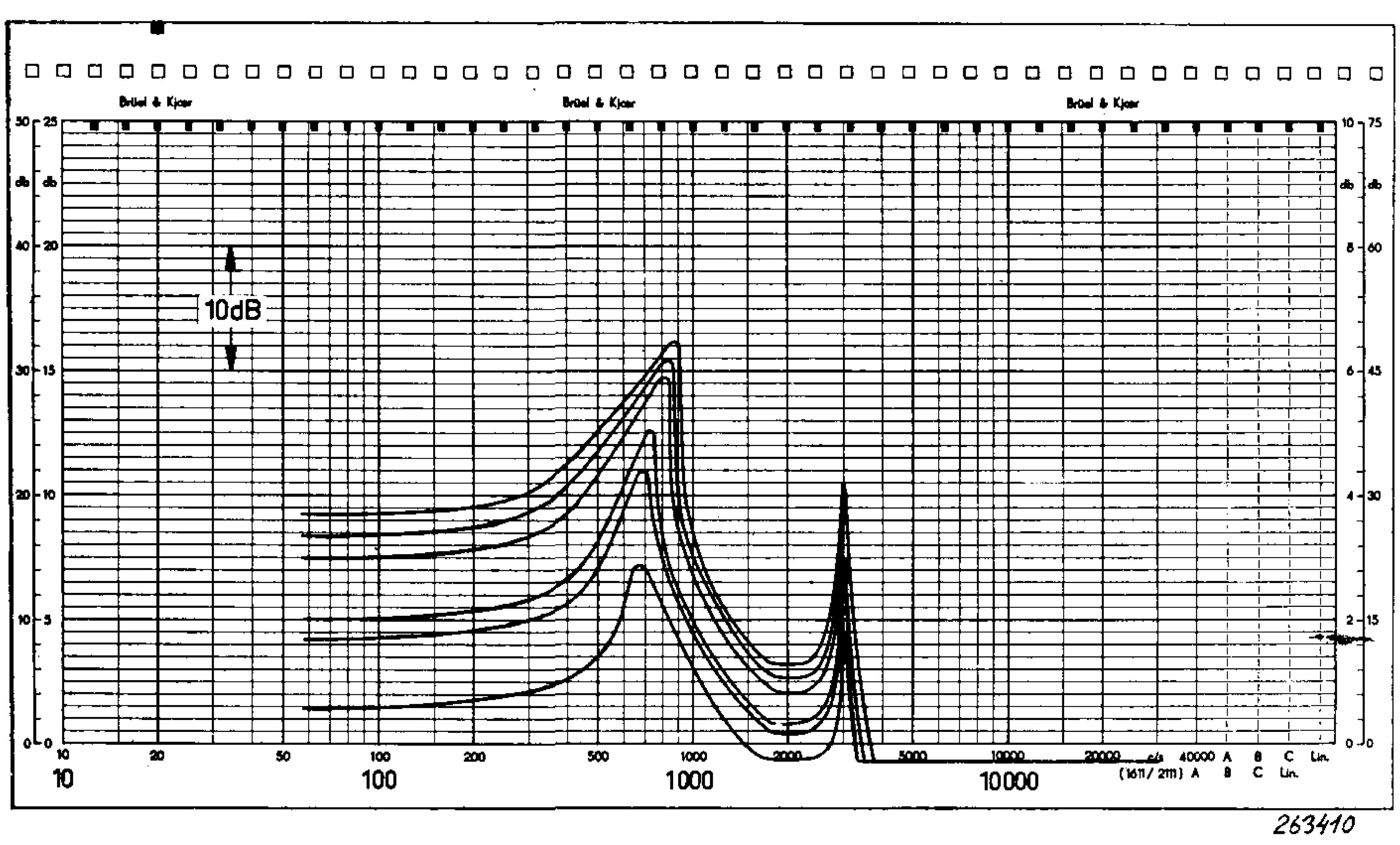
Fig. 30. Similar to Fig. 29 but the Q-value of the second circuit is here considerably smaller.

If the second resonance was "sharp" enough to only contain *one* harmonic of the first, the movement of the second mass would consist of a pure sinusoid superimposed upon the motion of the first mass, see Fig. 29. However, for small and reasonably large Q-values in the second circuit the sinusoid degenerates into what is commonly termed a "transient", i.e. a damped sinusoid, Fig. 30. The exact mathematical expression for the motion of the second mass is then extremely difficult to obtain and will, when finally written down, be very complicated, if at all possible, to use in practical cases. On the other hand, from experimental analogue studies as described here, some useful conclusions may be drawn.

As the "hardening" spring system seems to be the one producing the greatest amount of harmonics the non-linear circuit was chosen to be of this type. Furthermore, in the first instance the second, linear circuit was tuned to some frequency around three times the resonance frequency for small excitation levels of the "hardening" spring system. If now response curves are recorded for various excitation levels (input to the first circuit) a set of curves as shown in Fig. 31 is obtained. For the sake of clearity Fig. 31a shows response curves recorded when the input signal was sweeping upwards in frequency while in Fig. 31b the response curves for the same excitation levels but with the frequency sweep reversed are shown. Note the distinct differences in the shape of the curves which is easily explained from the results obtained earlier in this paper. The response curves have all been measured as the r.m.s.-value of the velocity of the second mass (voltage across the second capacitor), the system being excited at constant input velocity (input voltage) level.



a)



b)

Fig. 31. Frequency response curves (output r.m.s. velocity) recorded for various levels of excitation (constant input velocity).

a) Results obtained with the input signal sweeping upwards in frequency. Also photographs of the output wave-shape at certain frequencies as shown on the chart were taken.

b) Results obtained for reversed frequency sweep.

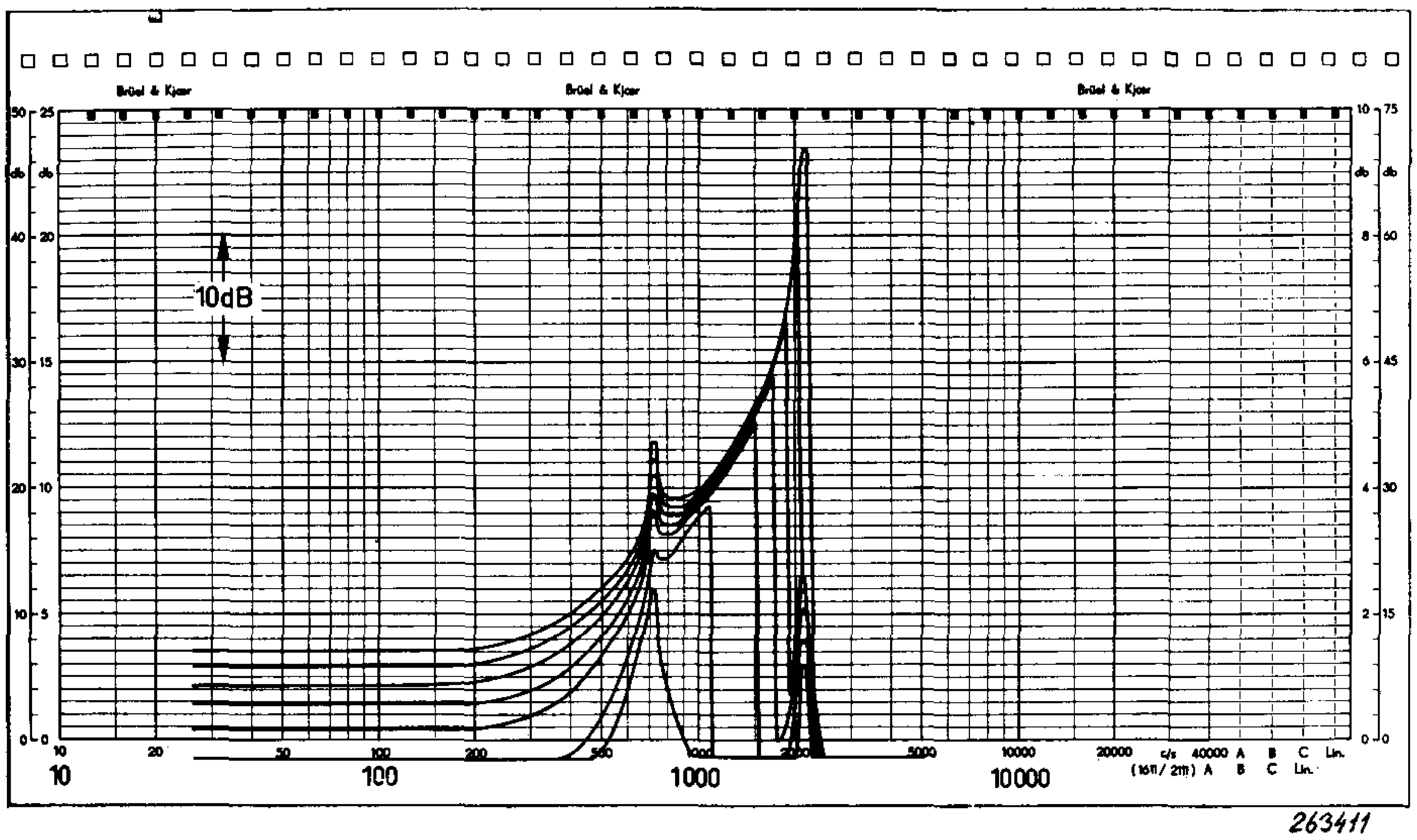


Fig. 32. Curves measured as in Fig. 31a) but with the second resonance moved slightly downwards in frequency.

In Fig. 32 the second resonance was moved slightly downwards in frequency to demonstrate how, by relatively great non-linearities and great excitation levels, the non-linearity of the first circuit can be brought to "cover" the second resonance.

To demonstrate the effect of the second circuit upon wave shape photographs were taken of the screen of an oscilloscope at certain input frequencies, Fig. 31a. Note the accentuation of the third harmonic due to the introduction of the second circuit.

Finally, some experiments were made with the input signal fixed at some 1000 c/s (see Fig. 31).

Firstly, a harmonic analysis of the output signal was made, the result of which can be seen in Fig. 33. As was to be expected the frequency component corresponding to the resonance frequency of the second circuit was strongly accentuated, its r.m.s. level actually being higher than that of the component corresponding to the input frequency.

Secondly, the resonance frequency of the second circuit was shifted so that it coincided with the seventh harmonic of the input signal, and the Q-value of the circuit varied. In Fig. 34 some photographs of the output wave-shapes are shown. The wave-shapes were obtained for different Q-values in the second circuit, and to show the actual composition of the signal two exposures were made in each case. The first exposure shows the output signal without introducing the second resonance, and the second exposure shows the result when the resonance was "in circuit". It is interesting to note the "modified" principle of superposition which is applicable in this case. Also the effect of damping the second resonance (reduction of its Q-value) is clearly demonstrated. The practical use of the "modified superposition principle" mentioned here will be discussed in later work which is in progress at the moment.

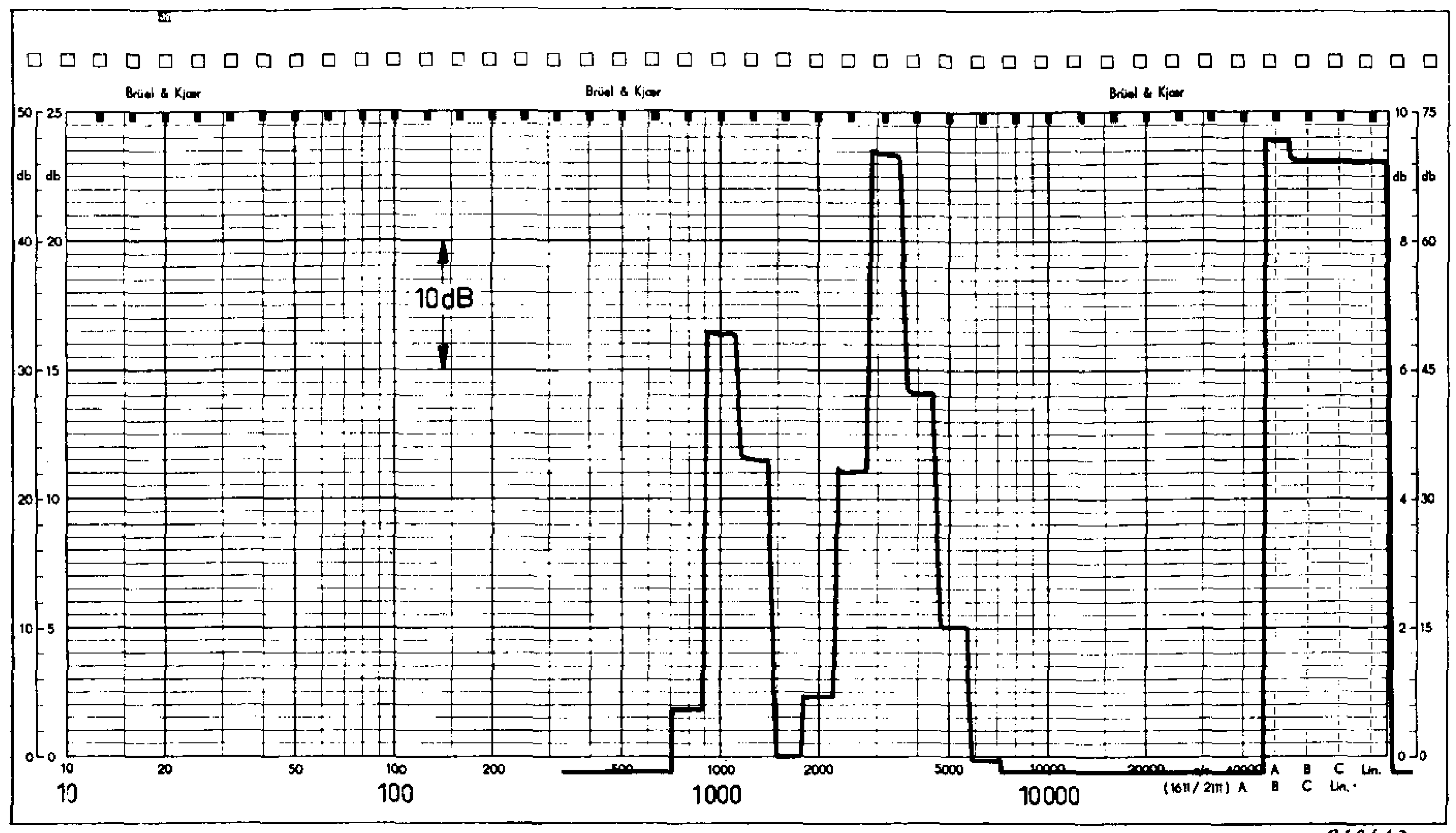
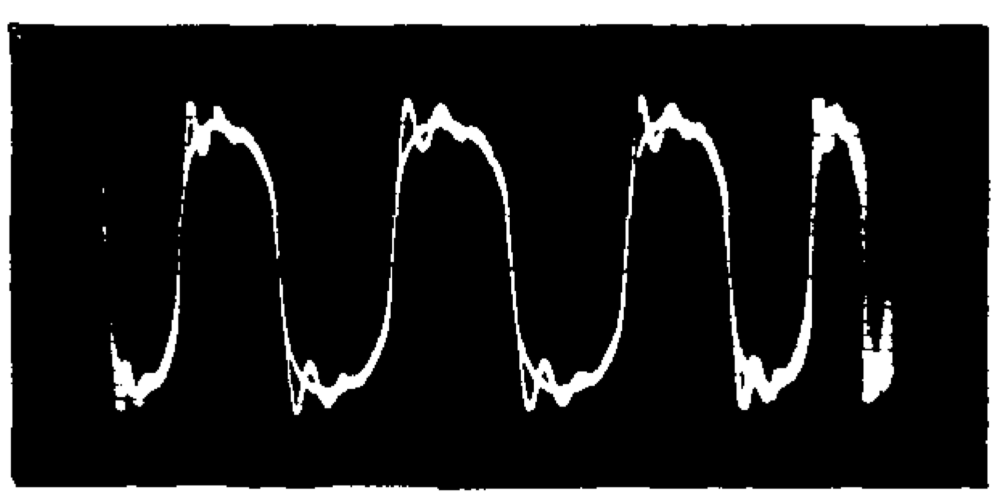
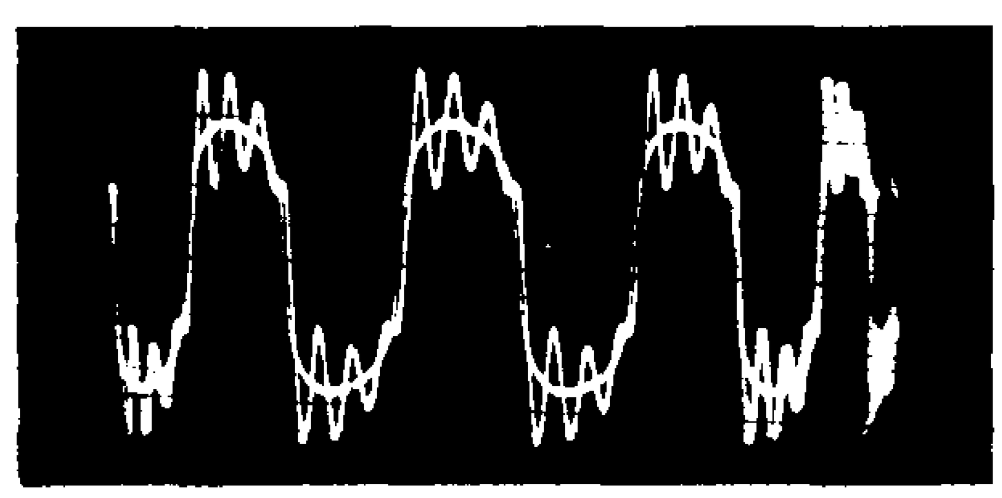


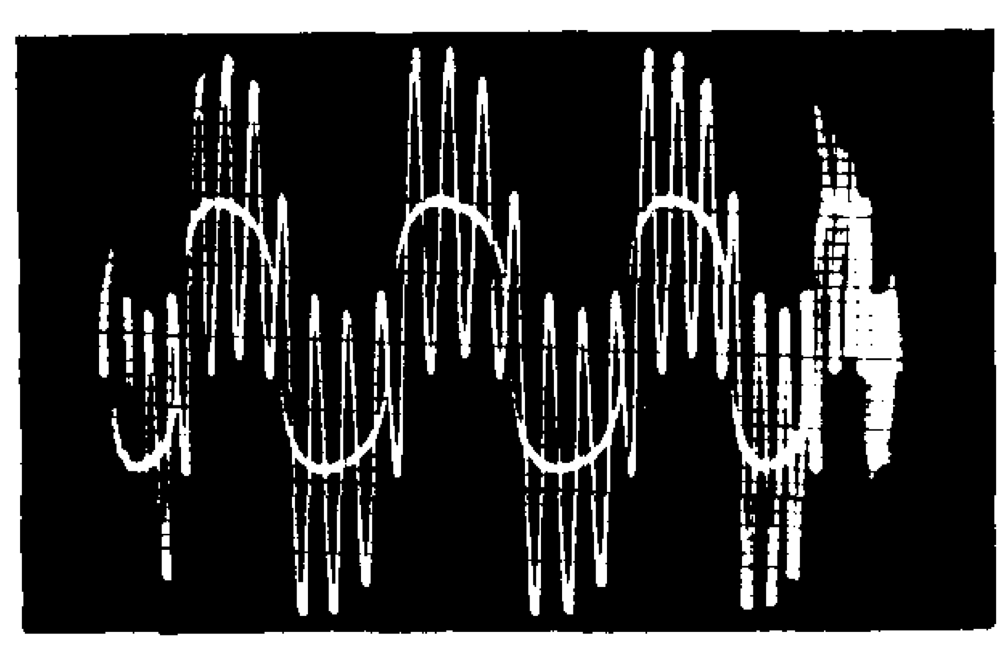
Fig. 33. 1/3 octave analysis of the output signal (velocity) with fixed input signal level and frequency. The measurements were carried out on the same system as used to obtain the curves in Fig. 31a).



Q = 3



Q = 7



Q = 25

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Fig. 34. Photographs of the output signal (velocity) wave-shape with various Q-values in the second resonant circuit.

Conclusion.

The aim of this article has been to demonstrate by means of analogue models, some of the effects produced by frequency dependent amplitude non-linearity. No efforts have been made to mathematically predict the results obtained as the expressions resulting from exact mathematical treatments are very difficult to handle and may be of little direct use to the practical engineer. However, a clear understanding of the physical effects might help to solve intricate design problems, and to understand "funny" test results.

A field where wave-shapes and spectra as discussed in this article commonly occur is the field of vibration testing. Very often the output from, for example a control accelerometer, shows waveshapes as illustrated under "Two Degrees-of-Freedom Systems" p. 27, or even more complicated waveshapes due to a greater number of "harmonically" related resonances. When this signal is used to control the motion of the shaker the actual vibration level at the specific test frequency will not at all be the one originally quoted for. Not much can be done in the way of outlining a "supreme" way of handling these problems.

It seems that if the control signal was filtered, this would ensure a correct input level at the specific test frequency, but results in an over-testing at higher frequencies. If no filtering is used the test level will be "too low" at the frequency of excitation. Thus the "correct" method of controlling the vibrator is more or less left to the judgement of the person in charge of the test programme, and must be decided upon with a view to the ultimate use of the equipment under test.

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